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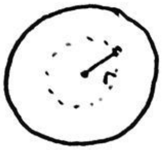
Lemma 1
It is well known fact that uniform eletric field induces eletric dipole, in center of the sphere, ~~of magnitude~~

$$\boxed{\vec{p} = 4\pi\varepsilon_0 R^3 \vec{E}}$$

Proof 1 Let's analyse eletric field of uniformly charged ball, which charge density is ρ.



pic. 1

For $r<R$ we have (using Gauss theorem)

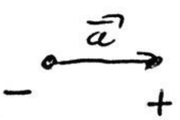
$$\vec{E} = \frac{\frac{4}{3} r^3 \pi \rho}{\varepsilon_0 4\pi r^2} \cdot \vec{r}$$

$$\vec{E} = \frac{\rho}{3\varepsilon_0}\vec{r}$$

Now, let's consider two fictious balls with charge densities of opposite signs and calculate eletric field in their intersection

$$\vec{E_x} = \vec{E_+} + \vec{E_-}$$

$$\vec{E_x} = \frac{\rho}{3\varepsilon_0}\left(\vec{r_+} - \vec{r_-}\right)$$



$$\boxed{\vec{E_x} = \frac{-\rho}{3\varepsilon_0} \cdot \vec{a}} \Rightarrow \text{it is uniform}$$

So if we take that $|a| \ll |R|$, and if $\frac{\rho \vec{a}}{3\varepsilon_0} = \vec{E}$, than field inside this new sphere will be zero, as desired. External field of this sphere will be same as if we considered this sistem of two charged balls as dipole. ~~because~~

$$\vec{p} = q \cdot \vec{a} = \frac{4}{3} R^3 \pi \cdot \rho\vec{a} -$$

$$\boxed{\vec{p} = 4\pi\varepsilon_0 R^3 \vec{E}} \quad \square$$

Now we must consider mutual influence of one ball on the other. Eletric field of one ball will be of order $\frac{kp}{L^3}$, and induced dipole will than be of order $p_0 \frac{R^3}{L^3}$ which is much smaller than p_0 ~~...~~

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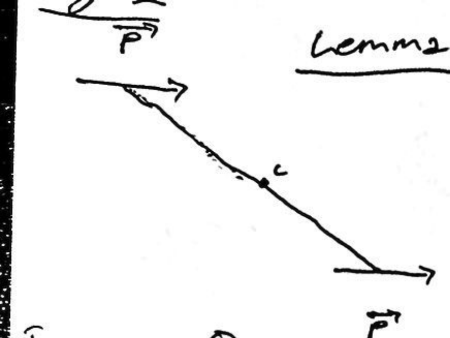
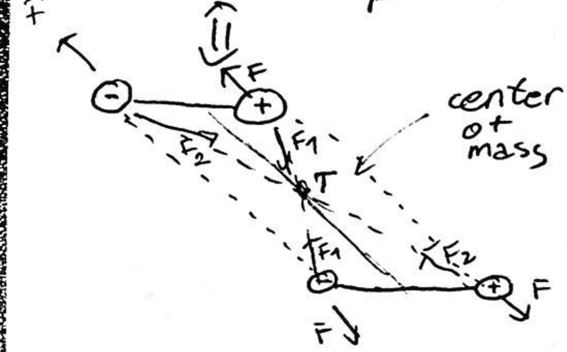
Lemma 2



If we analyse only moment of force of mutual interaction between primal induced dipoles we will get 0 (this is true for any position).

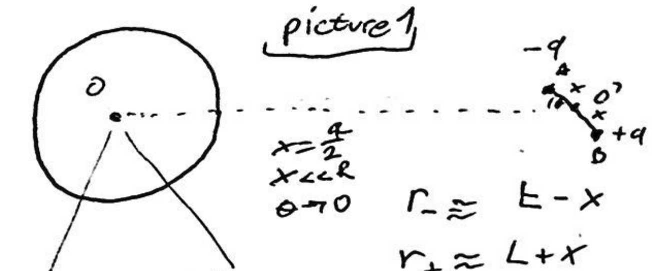
Here is why.



Proof 2

Moment of forces F will cancel each other, and forces F_1, F_2 have no moment. □

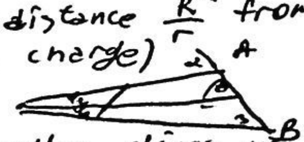
With all this in our mind we must analyse induced charges of one dipole to another one.

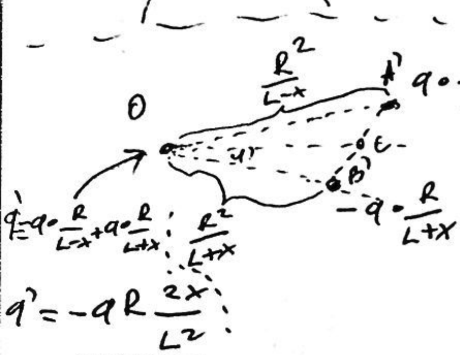


$r_- \approx L - x$

$r_+ \approx L + x$

We know that single charge q at distance r induces two charges in sphere; one of magnitude $q \cdot \frac{R}{r}$ at centre and another of magnitude $-q \cdot \frac{R}{r}$ in inversion of our starting charge (distance $\frac{R^2}{r}$ from centre, between center and charge)





$q' = -qR\frac{2x}{L^2}$

$\boxed{q' = -p\frac{R}{L^2}}$

picture 2

sin theorem for small angles gives us

$$\frac{\varphi_{1,2}}{x} = \frac{\theta}{L} \Rightarrow \varphi \ll \theta$$

Angle between $A'B'$ and OO' is $\alpha + \beta - \theta$ which is approximately θ.

Charge at O will be $q' = -p \cdot \frac{R}{L^2}$

Charge at A' will be $q \cdot \frac{R}{L-x}$ — picture 2

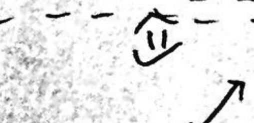
charge at B' will be $q \cdot \frac{R}{L+x}$

picture 3

We can rewrite $q_{A'}$ as $q \cdot \frac{R}{L} + \frac{pR}{2L^2}$ and $q_{B'}$ as $-q\frac{R}{L} + \frac{pR}{2L^2}$.

$|OC| = \frac{R^2}{L}$ $\quad |A'B'| = 2x \cdot \frac{R}{L}$ — picture 3

We can then put $\frac{pR}{2L^2}$ in c (these are small charges compared to $q \cdot \frac{R}{L}$). We get two dipoles then.



$$\vec{P_1} = \left(p \cdot \frac{R}{L^2}\right) \cdot \frac{R^2}{L} = p \cdot \frac{R^3}{L^3}$$

$$P_2 = \left(q \cdot \frac{R}{L}\right) \cdot \frac{Ra}{L^2} = p \cdot \frac{R^3}{L^3}$$

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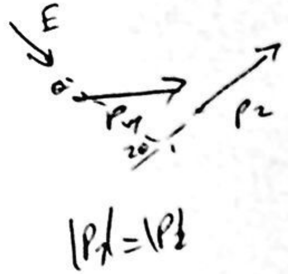
 Now external eletric field gives torque moment: $\vec{M} = \vec{p} \times \vec{E}$



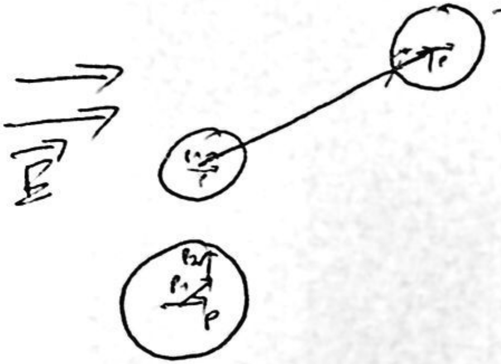
$$\vec{M} = \vec{p_1} \times \vec{E} + \vec{p_2} \times \vec{E}$$

$$\vec{M}' = p_1 E \sin\theta + p_2 E \sin 2\theta$$

$$M'_2 = 3\theta \cdot p' = \boxed{3\theta \cdot p \cdot \frac{R^3}{L^3} E}$$

We get absolutely ~~same~~ analogous thing when considering second sphere.

$$\boxed{M_1 = 2M' = 6p\frac{R^3}{L^3}E \cdot \theta}$$



Induced charges of dipoles p_1 and p_2 will be unhelpful because they will be really small (~~of ort~~ dipoles will be of order $p \cdot \frac{R^6}{L^6}$)

$$I\ddot{\theta} = -M_1$$

$$\frac{1}{2}ML^2\ddot{\theta} = -6p\frac{R^3}{L^3}E\theta$$

$\Downarrow$

$$T = 2\pi\sqrt{\frac{\frac{1}{2}ML^2}{24\pi\varepsilon_0 E^2 \frac{R^6}{L^5}}}$$

$$\boxed{T = \sqrt{\frac{\pi M L^5}{12\varepsilon_0 E^2 R^6}}}$$