Problem 1

Physics Cup

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We begin by calculating the polarizability of the construction. We will do this in two steps. First we consider the polarizability along the axis of the rod and then we calculate the polarizability orthogonal to the rod axis. We can break it up like this since electric fields can be added linearly and therefore we can 'split' the field into one component along the axis of the rod and one orthogonal component.

Let us start by denoting the polarizability of a sphere α. It is known that $\alpha = 4\pi\epsilon_0 R^3$. Next let us consider the two metal balls at a distance L, connected through a rod with negligible electrical effects, from each other with an electric field E_x along the axis of the rod. By symmetry the dipole moment of both of the spheres will be equal. Let us call it $\vec{p}$. The electrical field far away from a dipole is be calculated through

$$\vec{E}(\vec{r}) = \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{4\pi\epsilon_0 r^3}.$$

At a distance L from the dipole along the axis of the dipole moment the field is

$$\vec{E}(r) = \frac{2\vec{p}}{4\pi\epsilon_0 r^3}.$$

The dipole moment induced at one ball due to the external electric field $\vec{E}_x = E_x\hat{x}$ and the field from the other ball can be calculated through

$$p = \alpha(E_x + \frac{2p}{4\pi\epsilon_0 L^3}).$$

Solving this for p gives

$$p = \frac{\alpha E_x}{1 - \frac{2\alpha}{4\pi\epsilon_0 L^3}}$$

.

Now we can do similar calculations but where the external electric field $\vec{E}_y = E_y\hat{y}$ is orthogonal to the axis of the rod. In this case calculations are very similar except for the fact that the electric field around the dipole in the direction orthogonal to the dipole moment is

$$\vec{E}(r) = \frac{-\vec{p}}{4\pi\epsilon_0 r^3}.$$

And therefore the dipole moment instead becomes

$$p = \alpha(E_y - \frac{p}{4\pi\epsilon_0 L^3}),$$

which, solving for p, yields

$$p = \frac{\alpha E_y}{1 + \frac{\alpha}{4\pi\epsilon_0 L^3}}.$$

Adding these results together we get that

$$\vec{p} = \frac{2\alpha E_x}{1 - \frac{2\alpha}{4\pi\epsilon_0 L^3}}\hat{x} + \frac{2\alpha E_y}{1 + \frac{\alpha}{4\pi\epsilon_0 L^3}}\hat{y}$$

when we have the electric field

$$E = E_x\hat{x} + E_y\hat{y} = E(-sin\theta\hat{x} + cos\theta\hat{y}).$$

Note that we have multiplied both terms by 2 since there are two dipoles of equal dipole moment in the system, namely the two balls. The torque acting on a dipole is $\vec{\tau} = \vec{p} \times \vec{E}$ Plugging in the value of $\vec{p}$ into this equation gives us

$$\vec{\tau} = (\frac{2\alpha E_x E_y}{1 - \frac{2\alpha}{4\pi\epsilon_0 L^3}} - \frac{2\alpha E_x E_y}{1 + \frac{\alpha}{4\pi\epsilon_0 L^3}})\hat{z}.$$

Using $L >> R$ (and the fact that $\alpha = 4\pi\epsilon_0 R^3$) we can linearize this expression and get

$$\tau = \frac{6\alpha^2}{4\pi\epsilon_0 L^3} E_x E_y.$$

Now we plug in values for E_x and E_y and get

$$\tau = -\frac{6\alpha^2 E^2}{4\pi\epsilon_0 L^3}\frac{sin2\theta}{2}.$$

Which, since we have very small oscillations, is approximately

$$\tau = -\frac{6\alpha^2 E^2}{4\pi\epsilon_0 L^3}\theta.$$

Since $R << L$ we ignore the fact that the spheres are not point masses when calculating the moment of inertia around the centre of mass. Thus $I = 2M(\frac{L}{2})^2 = \frac{ML^2}{2}$. Since the moment of inertia is constant we have $\tau = I\ddot{\theta}$ and thus

$$\ddot{\theta} = -\frac{6\alpha^2 E^2}{4\pi\epsilon_0 I L^3}\theta = -\frac{12\alpha^2 E^2}{4\pi\epsilon_0 M L^5}\theta.$$

Th= is differential equation is the equation for simple harmonic motion with

$$\omega = \sqrt{\frac{12\alpha^2}{4\pi\epsilon_0 M L^5}} E,$$

or equivalently

$$\omega = \sqrt{\frac{48\pi\epsilon_0 R^6}{M L^5}} E$$

. Now we just need to plug this into $T = \frac{2\pi}{\omega}$ to get our expression for the period

$$T = \sqrt{\frac{\pi M L^5}{12\epsilon_0 R^6 E^2}}$$