Solution of Physics Cup 2020, Problem 1

JOHANES SUHARDJO

Due to the uniform electric field E, the metal sphere will be polarized with uniform polarization. There are many ways to find the polarization, one is to consider two spheres of uniform and opposite charge density separated by a small distance. It is well known that the electric field inside will be uniform, and this will cancel the external field. The electric field produced is $E = \rho a/3\epsilon_0$, with a the separation distance. Thus, the polarization (dipole moment per unit volume) is just $3\epsilon_0 E$.

Now, in the problem there are two spheres. The electric field of one will affect the other. However, the distance between the spheres is much larger compared with the radius. We then need to keep just some terms. The electric field of a uniformly polarized sphere is exactly like the electric field of a dipole outside the sphere. (This is because if we consider the first paragraph, the electric field of a uniformly charged sphere is exactly like a point charge at the center of the sphere.) The electric field of a dipole in polar coordinate is

$$\vec{E}_{dip} = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

The dipole moment induced by the uniform electric field is $4\pi\epsilon_0 R^3 E$, therefore it will produce electric field of the order $\beta^3 E$ at the other sphere, with $\beta = R/L$. This will again induced some surface charges on the sphere. (This time the electric field is not uniform, but since $\beta << 1$ we can approximate it to be uniform.) The induced polarization is then can also be assumed to be uniform. Next, the electric field produced by this new induced dipole at the other sphere can be neglected as it will be in the order of β^6 (if L is not much larger than R, we need to keep doing this). As a result, at one of the sphere we may only keep the external electric field and the electric field of a dipole produced by the other sphere. Likewise, for the induced charge, we only need to keep the charge induced by the external field, and the one induced by the dipole field.

Since the total charge is zero, the total force acting on the system must be zero. The oscillation must therefore be caused by torque. To determine which angle is the stable equilibrium, we can consider the potential energy of a dipole in electric field: the potential energy will be minimum if the dipole is in the same direction as the electric field. Let us call the angle between the rod and the external electric field by θ. When $\theta = 0$, the first dipole moments (induced by the external field) have the same direction as the external field and the second dipole moments (induced by the field of the first dipoles) also have the same direction as the external field, so this will be an equilibrium. When $\theta = \pi/2$, the system will also be in equilibrium, however, the electric field produced by the first dipole moments has the opposite direction with the external field, so the total dipole moment will slightly be smaller compared to when $\theta = 0$. The potential energy is smaller when the total dipole induced is bigger and thus, the stable equilibrium is when $\theta = 0$ ($\theta = \pi/2$ corresponds to unstable equilibrium).

We are ready now to calculate the period of small oscillation. When θ is small, the magnitude of the dipole field can be approximated by,

$$E_{dip} = \frac{2p}{4\pi\epsilon_0 r^3} = 2E\frac{R^3}{L^3}$$

with E the external electric field. The direction of the dipole field can be obtained the following way. The electric field in the θ direction is $E_\theta = E_r\theta/2$. So, the resultant electric field will make an angle $\theta/2$ with the rod or and angle $\phi = 3\theta/2$ to the external field. Now, the first dipole moment induced has the same direction with the external field, so no torque will be produced. The second dipole moment induced is making an angle ϕ with the external field, so the torque will be

$$\tau = -2Ep' sin\phi$$

Where the factor 2 is because there are two spheres and $p' = 4\pi\epsilon_0 R^3 E_{dip} = 8\pi\epsilon_0 ER^6/L^3$. Notice also that the electric field of a dipole is the same at an angle θ and $\pi - \theta$. So the equation of motion is

$$-\frac{24\pi\epsilon_0 E^2 R^6}{L^3}\theta = 2M\left(\frac{L}{2}\right)^2\frac{d^2\theta}{dt^2}$$

and finally

$$\omega^2 = \frac{48\pi\epsilon_0 E^2 R^6}{ML^5}$$

Additional Discussion

Following the first hint given, there should be no torque caused by dipole-dipole interaction due to Newton's third law. However, it seems that there is a torque if we approximate the field of a dipole to be uniform at the other sphere. This is because the electric field will make an angle $\phi = 3\theta/2$ with the dipole of the other sphere. The explanation to this must come from the fact that the electric field is not uniform, the outer part of the sphere must feel smaller electric field resulting zero total torque on the system. One question then keeps bothering me. If we approximate the dipole field of one sphere to be uniform at the other sphere, it will produce the wrong torque for dipole-dipole interaction. So, if this is the case, can we still approximate the second dipole moment as if it is induced by a uniform field? As the electric field is no longer uniform the polarization will not uniform as well, and the induced surface charge will slightly be different. How can we be sure that using this approximation will not give us the wrong torque caused by the external field just like in the dipole-dipole interaction case?

The difference of the dipole field is in the order of β^4, so the difference of the surface charge should also be in this order. This is small, but it is big enough to produce the wrong torque (for the dipole-dipole interaction). We need to show that this doesn't rise different torque for the torque by external field. To find exactly the surface charge distribution we can use method of images. Suppose there is an electric dipole at a distance s from a neutral conducting sphere with the direction of the dipole is making an angle α. The dipole can be seen as two opposite charges q and $-q$ separated by a small distance d. The distance of q to the center of the sphere will be $s - d\cos(\alpha)/2$ and the distance of $-q$ will be $s + d\cos(\alpha)/2$. Each will induce an image charge as to make an image dipole at a distance $b = R^2/s$. Let us call the direction of the image dipole by the angle β and the separation is d'. Then we must have

$$b + \frac{d'}{2}\cos\beta = \frac{R^2}{(s + \frac{d}{2}\cos\alpha)}$$

for the negative charge and

$$b - \frac{d'}{2}\cos\beta = \frac{R^2}{(s - \frac{d}{2}\cos\alpha)}$$

for the positive charge. Since d and d' are small we must have

$$d'\cos\beta = -\frac{R^2}{s^2} d\cos\alpha \tag{1}$$

We must also have from the triangle

$$\frac{d\sin\alpha}{s} = \frac{d'\sin\beta}{b} \tag{2}$$

Combining the two equations we will have $\beta = -\alpha$ and $d' = dR^2/s^2$. Notice that each charge will induce an image charge of different sign (from the two equations we need to get $\beta = \pi - \alpha$, but the direction of the image dipole is $-\alpha$ because of the sign)

Now the charge q will induced an image charge $-Rq/(s - d\cos\alpha/2)$ and charge $-q$ will induced an image charge $Rq/(s + d\cos\alpha/2)$. So, the image dipole will be

$$p' = \frac{R}{s} q d' = \frac{R^3}{s^3} p$$

Additionally we need a point charge at the position of the image dipole with charge $Q = -qRd\cos\alpha/s^2 = -pR\cos\alpha/s^2$. This is as one can easily see is because the image charges are different in magnitude. As the sphere is neutral another point charge $-Q$ should be put at the center of the sphere.

Having this result we may now calculate the surface charge distribution on the sphere. As one can easily check the surface will not be exactly the same as caused by uniform polarization case. (This time the dipole moment is slightly shifted from the center and there is also contribution of the two point charges Q and $-Q$.) However, we don't need to calculate the exact surface charge distribution to obtain the torque by the external field as it will be the same as if the force is acting on the image dipole and image charges. In the oscillation case, it is clear that $\theta = \alpha$. The torque on one sphere is

$$\tau = EQb\sin\theta - Ep'\sin 2\theta \approx -3Ep\frac{R^3}{L^3}\theta$$

Due to the fact that a torque produced by a uniform field to two point charges separated by a big distance is exactly the same with the torque produced to a dipole, it turns out that the torque will completely be the same. Interestingly if even the distance L is not much larger than R, the torque produced by the external field to the surface charges induced by the first dipole field will still be the same (as it being induced by uniform field). (We still need $L >> R$ though so that we can neglect the torque on smaller order charges.) Thus, we have shown that the same torque will apply