Physics Cup - Problem No 1

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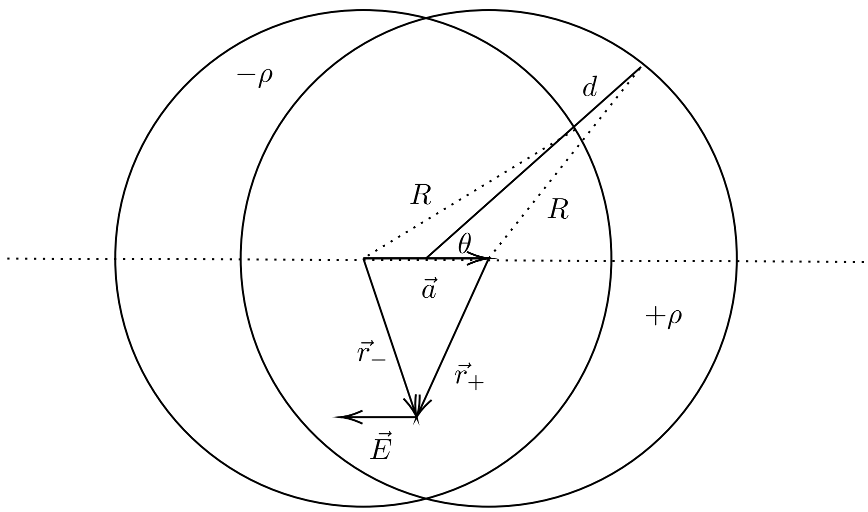
We consider the following charge configuration: two spheres with radius R with uniform volume charge density. One sphere has volume charge density $+\rho$, one sphere has volume charge density $-\rho$, the center of the two spheres are at the distance a ($a < R$) from each other.



We draw a line from the midpoint of the two centers forming an angle θ with $\vec{a}$, this line intersects the two spheres. Let the distance from the midpoint of the two centers and the intersections be d_1 and d_2 and the distance between the intersections is d. From the diagram, we have:

$$d_1^2 + \left(\frac{a}{2}\right)^2 - 2\frac{a}{2}d_1\cos\theta = d_2^2 + \left(\frac{a}{2}\right)^2 + 2\frac{a}{2}d_2\cos\theta = R^2$$

Then:

$$d_1^2 - d_2^2 = (d_1 + d_2)(d_1 - d_2) = a(d_1 + d_2)\cos\theta$$

Therefore:

$$d(\theta) = d_1 - d_2 = a\cos\theta$$

The electric field caused by each sphere inside itself can be easily derived from the Gauss' law:

$$\vec{E}_+ = \frac{\rho}{3\varepsilon_0}\vec{r}_+ \text{ and } \vec{E}_- = -\frac{\rho}{3\varepsilon_0}\vec{r}_-$$

The electric field caused by the two sphere at a point inside the overlapping region of the two spheres is:

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{3\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = -\frac{\rho}{3\varepsilon_0}\vec{a}$$

So the electric field caused by the two sphere inside the overlapping region is uniform.

When a is very small, this configuration is equivalent to a sphere with radius R with no charge inside, but have charges on the surface with surface charge density:

$$\sigma(\theta) = \rho d = \rho a \cos\theta = \sigma_0 \cos\theta \text{ with } \sigma_0 \equiv \rho a$$

So a charge distribution on the surface of a sphere in the form: $\sigma(\theta) = \sigma_0 \cos\theta$ generates an uniform electric field inside the sphere: $\vec{E} = -\frac{\sigma_0}{3\varepsilon_0}\hat{n}$, where $\hat{n}$ is the unit vector parallel with the symmetric axis of this charge distribution, pointing from the half with negative charge to the half with positive charge.

The electric field generated by this charge distribution outside the sphere is equivalent to the field generated by two spheres uniformly charged with volume charge density $+\rho$ and $-\rho$ with their centers at a very small distance a to each other (as proved above) such that: $\sigma_0 = \rho a$. Recall that the electric field generated by a uniformly charged sphere outside of it is similar to the field generated by a point charge with charge equal to the total charge of the sphere. Then, the electric field generated by this charge distribution outside the sphere is equivalent to the field generated by a dipole with dipole moment:

$$\vec{p} = Q\vec{a} = \rho\frac{4\pi R^3}{3}\vec{a} = \frac{4\pi R^3}{3}\sigma_0\hat{n}$$

Now we consider the interaction of an uniform external electric field $\vec{E}_0$ with this charge distribution. This interaction is equivalent to the interaction of the electric field with two spheres uniformly charged with volume charge density $+\rho$ and $-\rho$ with their centers at a very small distance a to each other (as proved above). Recall that the force caused by an uniform external electric field acting on a uniformly charged sphere is equivalent to the force acting by this field on a point charge with charge equal to the total charge of the sphere. Then, this interaction is equivalent to the interaction with a dipole with dipole moment: $\vec{p} = \frac{4\pi R^3}{3}\sigma_0\hat{n}$. Then, the torque acting on the sphere is:

$$\vec{\tau} = \vec{p} \times \vec{E}_0$$

Now we consider a neutral metal sphere is placed inside an external uniform electric field $\vec{E}_0$. At the electrostatic equilibrium state, the charges will arrange on its surface in a way such that the electric field inside the sphere is $\vec{0}$. From the result above, we can deduce that the charge on the surface are arranged in the form: $\sigma(\theta) = \sigma_0 \cos\theta$. The electric field inside the sphere is:

$$\vec{E}_{in} = \vec{E}_0 - \frac{\sigma_0}{3\varepsilon_0}\hat{n} = \vec{0}$$

$$\Rightarrow \sigma_0 = 3\varepsilon_0 E_0$$

Then the equivalent dipole moment that generates the field outside the sphere is:

$$\vec{p} = \frac{4\pi R^3}{3}\sigma_0\hat{n} = 4\pi\varepsilon_0 R^3 \vec{E}$$

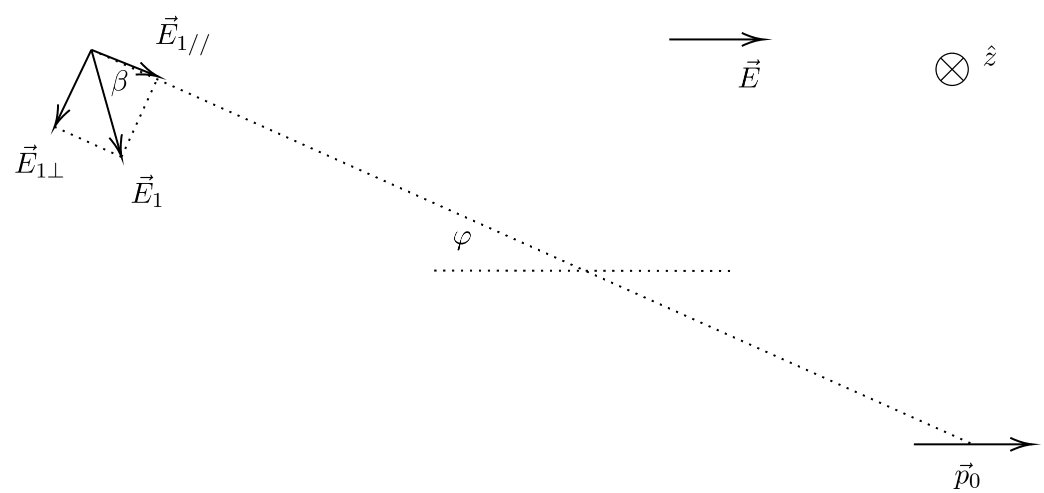
Now we consider the case of this problem. We will find the torque acting on this rigid body with respect to the center of mass when the rod is tilted with respect to the external electric field $\vec{E}$ a small angle φ. We assume that the metal sphere is a perfect conductor, which allows the charge inside to move so fast such that the charges will always be arranged in the distribution of the electrostatic equilibrium state. As the two spheres are connected with a dielectric rod, then the charges cannot move from one sphere to another.

We consider one metal sphere. Let the equivalent dipole of the charge distribution caused by the external electric field $\vec{E}$ and the electric field generated by the charges on the other sphere be $\vec{p}_0$ and $\vec{p}_1$, respectively.

As the electric field generated by $\vec{p}_0$ at distance R is at the same scale with E, the electric field generated by a dipole decrease by $distance^3$ and $L \gg R$, then the electric field generated by $\vec{p}_0$ at distance L would be much smaller than E. Therefore: $p_1 \ll p_0$.

Therefore, we can consider the electric field generated by the charges on the other sphere at the position of the sphere we are considering is the electric field of $\vec{p}_0$ at distance L. Since $L \gg R$, we can approximate this electric field to be uniform in the region of the sphere we are considering.

Let x be the direction of $\vec{E}$. Now we find the electric field generated by the charges on the other sphere at the position of the sphere we are considering ($\vec{E}_1$). We will consider only the first order approximation.



We have:

$$E_{1//} = \frac{p_0}{2\pi\varepsilon_0 L^3}\cos\varphi \text{ and } E_{1\perp} = \frac{p_0}{4\pi\varepsilon_0 L^3}\sin\varphi$$

$$E_1 = \frac{p_0}{4\pi\varepsilon_0 L^3}\sqrt{\sin^2\varphi + 4\cos^2\varphi} = \frac{p_0}{4\pi\varepsilon_0 L^3}\sqrt{4 - 3\sin^2\varphi} \approx \frac{p_0}{2\pi\varepsilon_0 L^3}$$

In the last approximation, we neglect the φ^2 term.

The angle between $\vec{E}_1$ and $\vec{E}_{//}$ is:

$$\beta \approx \tan\beta = \frac{\vec{E}_{1\perp}}{\vec{E}_{1//}} = \frac{\tan\varphi}{2} \approx \frac{\varphi}{2}$$

As $\vec{p}_1$ and $\vec{E}_1$ are parallel and in the same direction, then the angle between $\vec{p}_1$ and $\vec{E}$ is:

$$\alpha = \beta + \varphi = \frac{3\varphi}{2}$$

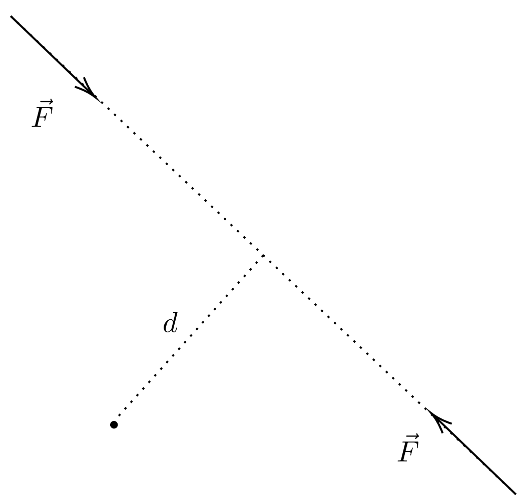
We have:

$$p_0 = 4\pi\varepsilon_0 R^3 E$$

$$p_1 = 4\pi\varepsilon_0 R^3 E_1 = \frac{2p_0 R^3}{L^3} = 8\pi\varepsilon_0 R^3 E \frac{R^3}{L^3}$$

Since the force between two point charges are parallel with the line connecting them, then the total torque acting on two point charge interacting with each other with respect to an arbitrary point is:

$$\tau = Fd - Fd = 0$$



So the total torque caused by the interaction between the charges inside the system is $\vec{0}$. The torque that pulls the system back to the equilibrium state is the torque caused by the external electric field on the the charges on the surface metal spheres. As stated above, this torque is equivalent to the torque acting on the equivalent dipole.

The torque of $\vec{E}$ acting on $\vec{p}_0$ is:

$$\vec{\tau}_0 = \vec{p}_0 \times \vec{E} = 4\pi\varepsilon_0 R^3(\vec{E} \times \vec{E}) = \vec{0}$$

The torque of $\vec{E}$ acting on $\vec{p}_1$ is:

$$\vec{\tau}_1 = \vec{p}_1 \times \vec{E} = -8\pi\varepsilon_0 R^3 E^2 \frac{R^3}{L^3} \sin\alpha \hat{z} \approx -8\pi\varepsilon_0 R^3 E^2 \frac{R^3}{L^3} \alpha \hat{z} = -12\pi\varepsilon_0 R^3 E^2 \frac{R^3}{L^3} \varphi \hat{z}$$

The total torque of $\vec{E}$ acting on the charges on the surface of one metal sphere is:

$$\vec{\tau} = \vec{\tau}_0 + \vec{\tau}_1 = -12\pi\varepsilon_0 R^3 E^2 \frac{R^3}{L^3} \varphi \hat{z}$$

Similarly, the torque of $\vec{E}$ acting on the other sphere is also $\vec{\tau} = -12\pi\varepsilon_0 R^3 E^2 \frac{R^3}{L^3} \varphi \hat{z}$.

As $L \gg R$, the moment of inertia of this rigid body with respect to axis passing through the center of mass is:

$$I \approx 2M \left(\frac{L}{2}\right)^2 = \frac{ML^2}{2}$$

We have:

$$I\ddot{\varphi} = -12\pi\varepsilon_0 R^3 E^2 \frac{R^3}{L^3} \varphi - 12\pi\varepsilon_0 R^3 E^2 \frac{R^3}{L^3} \varphi$$

Then:

$$\ddot{\varphi} + \frac{48\pi\varepsilon_0 E^2 R^6}{ML^5} \varphi = 0$$

The period of the small oscillation around the stable equilibrium state is:

$$P = 2\pi \sqrt{\frac{ML^5}{48\pi\varepsilon_0 E^2 R^6}} = \sqrt{\frac{\pi M L^5}{12\varepsilon_0 E^2 R^6}}$$