Problem NO 1

1. Polarization of conducting sphere.

Consider : the Laplace Equation

$$\nabla^2 \varphi = 0$$

And we have the boundary condition

$$r \to \infty, \quad \varphi \to -Er\cos\theta \quad ; \quad r \to R \quad \frac{\partial \varphi}{\partial \theta} \to 0$$

According to the Legendre polynomials. we have the form of the solution

$$\varphi = -Er\cos\theta + \frac{A}{r^2}\cos\theta \qquad A = ER^3$$

i.e. $$\varphi = -Er\cos\theta + \frac{ER^3}{r^2}\cos\theta$$

The second part is similar to the electric field generated by electric dipoles.

So. the ball is equivalent to the electric dipole

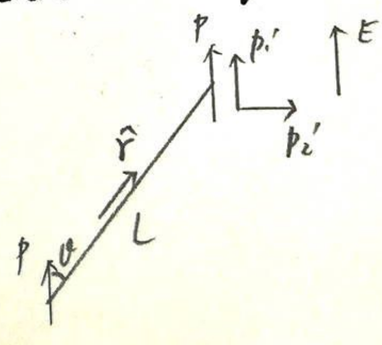
$$\vec{p} = 4\pi\varepsilon_0 R^3 \vec{E}$$

2. The motion of one ball.

Since $L \gg R$, and the interaction force between two balls $F \propto \frac{p^2}{L^4} \propto \frac{R^6}{L^4}$, it's necessary to consider the first order induction between the balls.

As the electric field generated by electric dipoles is.

$$\varphi_1 = \frac{\vec{p} \cdot \hat{r}}{4\pi\varepsilon_0 r^2},$$



We can obtain that.

$$\vec{E}_1 = -\nabla\varphi_1 = \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{4\pi\varepsilon_0 r^3}$$

So, $$p_1' = \frac{R^3}{L^3} p(3\cos^2\theta - 1) \qquad p_2' = \frac{R^3}{L^3} p \cdot 3\cos\theta\sin\theta$$

Consider the part from the center of mass to one ball.

$\vec{M}_p^{(e)}$: ~~ex~~ external electric field to the dipole

$\vec{M}_{pp}$: dipole to dipole

$\vec{F}_{pp}$: the force of dipole to dipole

$\vec{M}_{p'}^{(e)}$: external electric field to the induction dipole.

It's clear that the total torque is

$$\vec{M}_t = \vec{M}_p^{(e)} + \vec{M}_{pp} + \left(\frac{L}{2}\hat{r}\right) \times \vec{F}_{pp} + \vec{M}_{p'}^{(e)}$$

$$\vec{M}_p^{(e)} = \vec{p} \times \vec{E} = 0$$

$$\vec{M}_{pp} = \vec{p} \times \vec{E}_1 = \frac{3p^2 \sin\theta \cos\theta}{4\pi \varepsilon_0 L^3}\hat{z} \qquad (\hat{z} = \hat{r} \times \hat{\theta})$$

$$\vec{F}_{pp} = \vec{p} \times \nabla \vec{E}_1 = \frac{3p^2}{4\pi\varepsilon_0 L^4}\left[(\hat{p}\cdot\hat{r})\hat{p} + (\hat{p}\cdot\hat{r})\hat{p} + (\hat{p}\cdot\hat{p})\hat{r} - 5(\hat{p}\cdot\hat{r})(\hat{p}\cdot\hat{r})\hat{r}\right]$$

$$= \frac{3p^2}{4\pi\varepsilon_0 L^4}(2\cos\theta\,\hat{p} + \hat{r} - 5\cos^2\theta\,\hat{r})$$

$$\vec{M}'_{pp} = \frac{L}{2}\hat{r} \times \vec{F}_{pp} = -\frac{3p^2 \sin\theta\cos\theta}{4\pi\varepsilon_0 L^3}\hat{z}$$

$$\vec{M}_{p'}^{(e)} = (\vec{p}_1' + \vec{p}_2') \times \vec{E} = \vec{p}_2' \times \vec{E} = -\frac{R^3}{L^3} pE \cdot 3\cos\theta\sin\theta\,\hat{z}$$

So, we can obtain the equation from angular momentum theorem.

$$M \cdot \left(\frac{L}{2}\right)^2 \ddot{\theta} = -\frac{3R^3}{L^3} pE\cos\theta\sin\theta$$

when $\theta \ll 1$, it turns to

$$M\left(\frac{L}{2}\right)^2 \ddot{\theta} = -\frac{3R^3}{L^3} pE\,\theta$$

So. $T = 2\pi\sqrt{\dfrac{M(L/2)^2}{3R^3pE/L^3}} = 2\pi\sqrt{\dfrac{ML^5}{48\pi\varepsilon_0 ER^6}} = \sqrt{\dfrac{\pi ML^5}{12\varepsilon_0 ER^6}}$

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