Physics Cup - Problem 1

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Due to the very strong magnetic field, the ball gets saturated, i.e it reaches its maximum magnetization. After we turn off the external field, the ball is still magnetized. The goal is to find the magnetization, which we define as the magnetic dipole moment per unit volume. We will do this by making analogies between magnetic and electric fields. First, we define the electric diplacement:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \tag{1}$$

where $\mathbf{P}$ is the polarization. We will also need the magnetic field $\mathbf{H}$ as a function of the magnetic induction $\mathbf{B}$:

$$\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right), \tag{2}$$

where $\mathbf{M}$ is the magnetization. We then write down Maxwell's equations in matter:

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot \mathbf{D} = \rho, \\
&\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \\
&\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\
&\boldsymbol{\nabla} \times \mathbf{H} = \mu_0 \left(\mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \right),
\end{aligned}$$

where ρ is the charge density, and j is the current density. As all fields are static and there are no currents or charges, these equations have the form:

$$\begin{aligned}
&\boldsymbol{\nabla} \cdot \mathbf{D} = 0, \\
&\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \\
&\boldsymbol{\nabla} \times \mathbf{E} = 0, \\
&\boldsymbol{\nabla} \times \mathbf{H} = 0.
\end{aligned}$$

If we look closely at these equations, we can see that one can make the correspondences $\mathbf{E}$ to $\mu_0 \mathbf{H}$ and $\frac{\mathbf{D}}{\epsilon_0}$ to $\mathbf{B}$ (we will see later why we should include the factors μ_0 and $\frac{1}{\epsilon_0}$). Thus, using Eq.(1) and Eq.(2) we have another correspondence, namely between $\frac{\mathbf{P}}{\epsilon_0}$ and $\mu_0 \mathbf{M}$. This means that we can look at a similar problem with electric fields.

Suppose that we have a metallic sphere with radius R in an external homogeneous electric field $\mathbf{E_{ext}}$. We can find the electric field due to the polarized sphere by superposing two balls with uniform charge densities ρ and $-\rho$ with their centers at a small distance a. Using Gauss's law, we can find the electric field due to the positively charged ball. Using a surface with radius r around the center, one obtains the following relations:

$$E4\pi r^2 = \frac{\rho\frac{4}{3}\pi r^3}{\epsilon_0}$$
$$\implies \mathbf{E} = \frac{\rho\vec{r}}{3\epsilon_0}.$$

After we superpose the fields from the positive and negative balls, we obtain the inner electric field due to the polarized sphere:

$$\mathbf{E_{sph}} = \frac{\rho\vec{a}}{3\epsilon_0}.$$

The whole electric field inside the sphere will then be

$$\mathbf{E_{in}} = \mathbf{E_{ext}} - \frac{\rho\vec{a}}{3\epsilon_0}. \tag{3}$$

By definition, the dipole moment of the sphere is

$$\mathbf{p} = \rho\frac{4}{3}\pi R^3\vec{a} = \frac{4}{3}\pi R^3\mathbf{P}.$$

From this, it follows that Eq.(3) becomes

$$\mathbf{E_{in}} = \mathbf{E_{ext}} - \frac{\mathbf{P}}{3\epsilon_0}.$$

Going back to the magnetic field problem, we use the analogies made earlier:

$$\mu_0\mathbf{H_{in}} = \mu_0\mathbf{H_{ext}} - \frac{\mu_0\mathbf{M}}{3}.$$

Here, we see that the additional factors noted earlier were needed due to dimensional analysis. For the rest of the solution we will use $\mathbf{H}$ instead of $\mathbf{H_{in}}$. Now, we use the fact that $\mathbf{B_{ext}} = \mu_0\mathbf{H_{ext}}$ and $\mathbf{B} = \mu_0\left(\mathbf{H} + \mathbf{M}\right)$ and obtain

$$\mathbf{B} = 3\mathbf{B_{ext}} - 2\mu_0\mathbf{H}. \tag{4}$$

As we turn off the external magnetic field, we use that $\mathbf{B_{ext}} = \mathbf{0}$. Then, on the hysteresis graph we draw a straight line, which crosses the origin and has a slope -2 (see Fig.(1)). The intersection points give us the values of $\mathbf{H}$ and $\mathbf{B}$

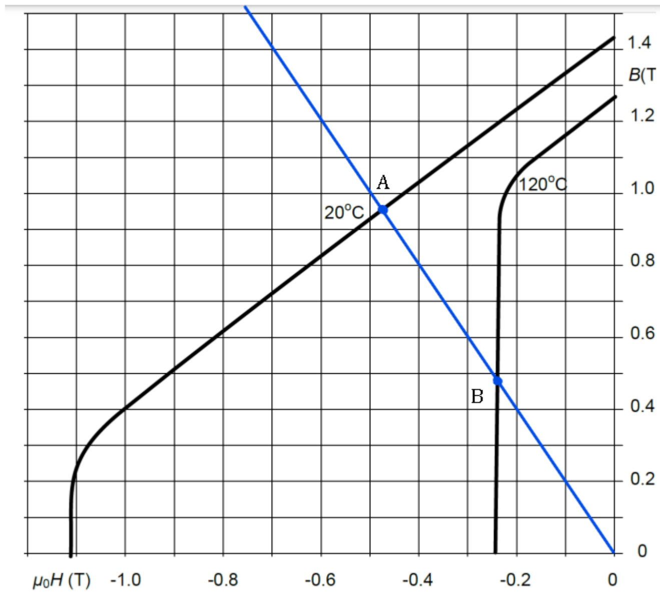

Figure 1: Intersection points A and B of the function (4) (the blue line) when $\mathbf{B_{ext}} = 0$ with the hysteresis curve.

for the two temperatures:

$$
\begin{aligned}
&B(20^\circ\mathrm{C}) \approx 0.95\,\mathrm{T},\\
&\mu_0 H(20^\circ\mathrm{C}) \approx -0.47\,\mathrm{T},\\
&B(120^\circ\mathrm{C}) \approx 0.48\,\mathrm{T},\\
&\mu_0 H(120^\circ\mathrm{C}) \approx -0.24\,\mathrm{T}.
\end{aligned}
$$

We can then find the magnetization of the ball for the different temperatures using $\mathbf{B} = \mu_0\left(\mathbf{H} + \mathbf{M}\right)$. By definition, the magnetic dipole moment is $\mathbf{p_m} = \frac{4}{3}\pi R^3 \mathbf{M}$. After plugging in the numeric values, we can easily find the decrease in the magnetic dipole moment due to the heating:

$$\boxed{\Delta p_m \approx 2.35\,\mathrm{A}\cdot\mathrm{m}^2}$$