Physics Cup 2022 Problem 1

Eddie Chen Zhening Li

November 2021

Lemma 1. *If a magnetic sphere is placed in an external uniform magnetic field* $\mathbf{B}_0$ *pointing in the z-direction, the sphere will become uniformly magnetized with uniform* $\mathbf{B}$ *and* $\mathbf{H}$ *inside the ball. They point in the direction of* $\mathbf{B}_0$ *and satisfy*

$$\mathbf{B} + 2\mu_0\mathbf{H} = 3\mathbf{B}_0. \tag{1}$$

Proof. First, note that

$$\begin{aligned}
\mathbf{H} &= \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \\
\nabla \times \mathbf{H} &= \frac{1}{\mu_0}\nabla \times \mathbf{B} - \nabla \times \mathbf{M} \\
&= \mathbf{J} + \epsilon_0\dot{\mathbf{E}} - \mathbf{J}_b \\
&= \mathbf{J}_f \\
\nabla \times \mathbf{H} &= \mathbf{0}.
\end{aligned} \tag{2}$$

We've used the fact that $\dot{\mathbf{E}} = \mathbf{0}$ (the system is static) and free current density $\mathbf{J}_f = \mathbf{0}$.

Recall also Gauss's law for magnetism:

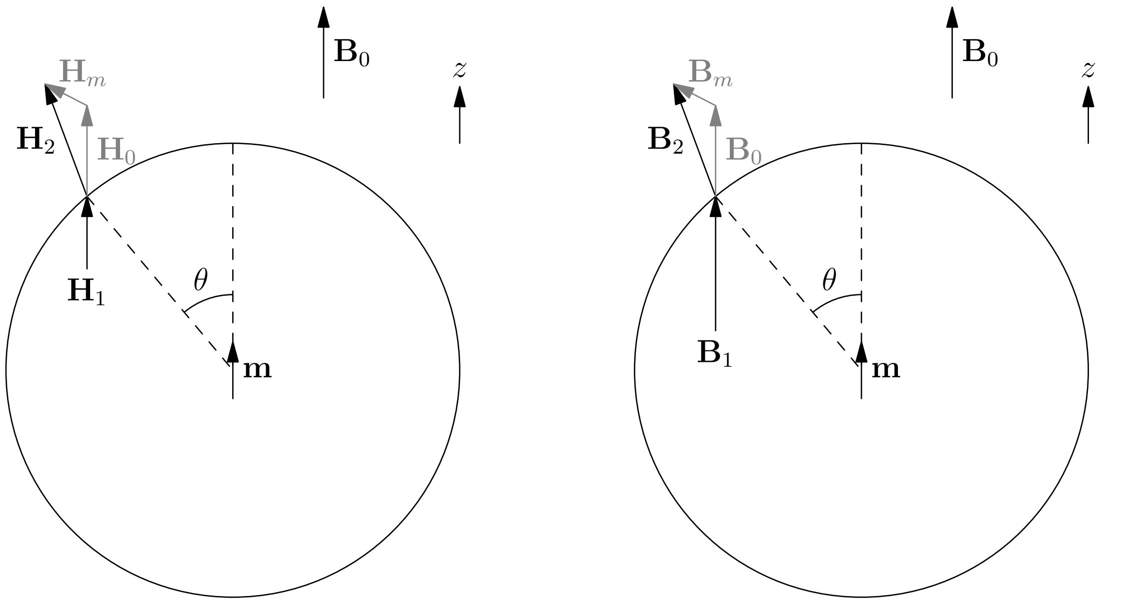
$$\nabla \cdot \mathbf{B} = \mathbf{0}. \tag{3}$$

In addition to (2) and (3), there is a direct relationship between $\mathbf{H}$ and $\mathbf{B}$ determined by the material's properties within the ball and given by $\mathbf{H} = \mathbf{B}/\mu_0$ outside. We also have boundary conditions that the fields tend to the externally applied field $\mathbf{B}_0$ at infinity. Thus, all of these conditions let us solve for $\mathbf{B}$ and $\mathbf{H}$.

A solution is described as follows.

1. In the region $r < R$, the B-field and H-field are uniform and point in the z-direction.

2. In the region $r > R$, the B-field is the superposition of the externally applied field $\mathbf{B}_0$ and the field $\mathbf{B}_m$ of a magnetic dipole $\mathbf{m}$ at $r = 0$ pointing in the z-direction.



(a) H-field at the surface of the ball

(b) B-field at the surface of the ball

Figure 1: Metal ball magnetized in a magnetic field

It is clear that (3) and (2) are satisfied in the region $r < R$ and in the region $r > R$. Also, the boundary conditions are clearly satisfied as the field due to the dipole $\mathbf{m}$ tends to zero at infinity. It now remains to check that the equations are satisfied at the interface between the sphere and the vacuum.

(2) translates to $H_\parallel$ is continuous across the interface. In other words, for a point on the interface at an angle θ from the z-direction (Figure 1a),

$$\begin{aligned}
\mu_0 H_{\parallel 1} &= \mu_0 H_{\parallel 2} = B_{\parallel 2} \\
\mu_0 H \sin\theta &= B_0 \sin\theta + B_{m\parallel} && (H := H\text{-field inside the ball}) \\
\mu_0 H \sin\theta &= B_0 \sin\theta - \frac{\mu_0 m \sin\theta}{4\pi R^3} \\
\mu_0 H &= B_0 - \frac{\mu_0 m}{4\pi R^3}. && (4)
\end{aligned}$$

(3) translates to $B_\perp$ is continuous across the interface. In other words, for a point on

the interface at an angle θ from the z-direction (Figure 1b),

$$\begin{aligned}
B_{\perp 1} &= B_{\perp 2} \\
B\cos\theta &= B_0\cos\theta + B_{m\perp} && (B := B\text{-field inside the ball}) \\
B\cos\theta &= B_0\cos\theta + \frac{\mu_0 m\cos\theta}{2\pi R^3} \\
B &= B_0 + \frac{\mu_0 m}{2\pi R^3}. && (5)
\end{aligned}$$

Eliminating m in (4) and (5) gives

$$B + 2\mu_0 H = 3B_0,$$

as desired.

Note that, to find the solution to the entire B- and H-fields, one determines (H, B) (the fields inside the ball) using (1) and the relationship between B and H as given by the properties of the ball's material (e.g. a hysteresis curve). Then m can be found using either (4) or (5), from which the fields outside the ball are determined.

□

Now, consider the ball in question. It is placed in a large external magnetic field that allows it to achieve saturation, so that it follows the hysteresis curve when the external field is removed. When the external field is completely removed ($B_0 = 0$), we have

$$B + 2\mu_0 H = 0$$

according to Lemma 1. On the graph, this corresponds to the line l with slope -2 through the origin (Figure 2).

The intersection of the hysteresis curve with l gives the (H, B) of the ball after the external applied field has been removed. This gives $B = 0.95\,\mathrm{T}$ at $20\,^\circ\mathrm{C}$ (see Figure 2). Then, when the ball is heated to $120\,^\circ\mathrm{C}$, it descends along l until it reaches the hysteresis curve corresponding to $120\,^\circ\mathrm{C}$, where the magnetic flux density becomes $B' = 0.48\,\mathrm{T}$ (see Figure 2). So the decrease in the magnetic flux density inside the ball is

$$B - B' = 0.47\,\mathrm{T}.$$

Since $B + 2\mu_0 H = 0$ and $H = B/\mu_0 - M$, we get $B + 2\mu_0(B/\mu_0 - M) = 0$, or

$$\begin{aligned}
3B - 2\mu_0 M &= 0 \\
M &= \frac{3}{2\mu_0}B.
\end{aligned}$$

Thus, the decrease in magnetization of the ball is

$$M - M' = \frac{3}{2\mu_0}(B - B') = 5.6 \times 10^5\,\mathrm{A/m}.$$

Finally, since magnetic dipole moment $m = VM = \frac{4}{3}\pi R^3 M$ for the ball, we obtain the following for the decrease in its magnetic dipole moment:

$$m - m' = \frac{4}{3}\pi R^3 (M - M') = 2.4\,\mathrm{A\,m^2}.$$

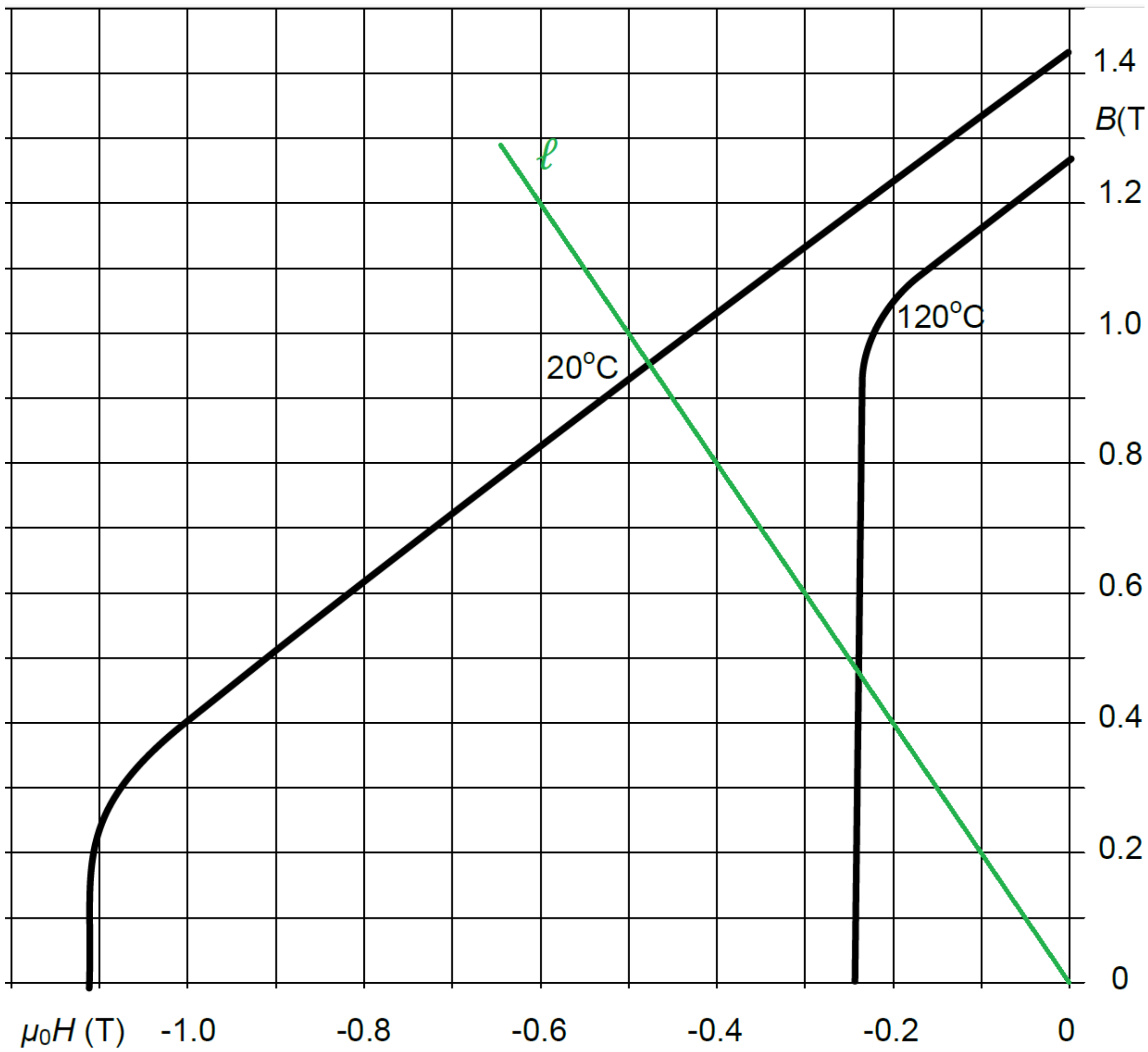


Figure 2: Hysteresis curves with the line l corresponding to $B_0 = 0$.