The solution to Problem No 2

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To make the duration of a loop as short as possible, the acceleration of the boy should always keep μg .

To make the calculation more clear, we set the unit of time as $\tau = \sqrt{\frac{a}{\mu g}}$, and omit all dimensions in following formulae.

Suppose the value of velocity is a function of the direction of the velocity $v(\theta)$, in the Natural Coordinate system:

$$(\frac{v^2(\theta)}{\rho})^2 + (\frac{\mathrm{d}v(\theta)}{\mathrm{d}t})^2 = 1 \quad \ldots\ldots \text{(1)}$$

ρ is the radius of curvature of the orbit. And replace $\mathrm{d}t$ with $\frac{\rho\,\mathrm{d}\theta}{v}$:

$$\frac{v^2}{\rho^2}[v^2(\theta) + (\frac{\mathrm{d}v(\theta)}{\mathrm{d}\theta})^2] = 1 \quad \ldots\ldots \text{(2)}$$

It’s easy to find that if the orbit becomes k times greater while velocity becomes $\sqrt{k}$ times greater, the equation remains correct, and the duration becomes $\sqrt{k}$ times greater as well. This means we should make the orbit as ‘small’ as possible. So the orbit must pass the vertexes of the triangle house.

Finally, considering the symmetry of the system, the angle between the orbit tangent at the vertex and the side of the house should be $\frac{\pi}{3}$.

As a conclusion, the total duration of a loop should be:

$$T = 3\int \mathrm{d}t = 3\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\rho\,\mathrm{d}\theta}{v} = 3\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sqrt{v(\theta)^2 + v^{'}(\theta)^2}\,\mathrm{d}\theta \quad \ldots\ldots \text{(3)}$$

And with the constraint of the house's size:

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} v\cos\theta\sqrt{v(\theta)^2 + v^{'}(\theta)^2}\,\mathrm{d}\theta = 1 \quad \ldots\ldots \text{(3)}$$

Now, suppose y as the velocity component parallel to the side of the triangle house, and x as the vertical velocity component. Then rewrite the formula(3)&(4) in the cartesian coordinates in this velocity space:

$$T = 3\int_{-x_0}^{x_0} \sqrt{1 + y'(x)^2}\,\mathrm{d}x \quad \ldots\ldots \text{(5)}$$

$$\int_{-x_0}^{x_0} y\sqrt{1 + y'(x)^2}\,\mathrm{d}x = 1 \quad \ldots\ldots \text{(6)}$$

Euler Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y^{'}}\right) - \frac{\partial f}{\partial y} = 0 \quad \ldots\ldots \text{(7)}$$

$$f = (1 - \lambda y)\sqrt{1 + y'^2} \quad \ldots\ldots \text{(8)}$$

And the solution of this equation is well known as the catenary :

$$y = \xi\cosh\frac{x}{\xi} + \frac{1}{\lambda} \quad \ldots\ldots \text{(9)}$$

$$y(x_0) = -\frac{\sqrt{3}}{3}x_0 \quad \ldots\ldots \text{(10)}$$

Now, imagine there is a 'string' hang on two 'sticks' in the velocity space:

$$y = \frac{\sqrt{3}}{3}x \quad \ldots\ldots \ (11)$$

$$y = -\frac{\sqrt{3}}{3}x \quad \ldots\ldots \ (12)$$

Obviously, the ends of the string must be perpendicular to the sticks:

$$y'(x_0) = \sqrt{3} \quad \ldots\ldots \ (13)$$

Combine the formula(6)(9)(10)(13):

$$\xi = \sqrt{\frac{1}{\operatorname{arsh}\sqrt{3} + 2\sqrt{3}}} = 0.457 \quad \ldots\ldots \ (14)$$

Combine the formula(5)(14):

$$T = 3\xi \int_{-\operatorname{arsh}\sqrt{3}}^{\operatorname{arsh}\sqrt{3}} \cosh x \cdot \mathrm{d}x = 6\sqrt{3}\xi = \sqrt{\frac{108}{\operatorname{arsh}\sqrt{3} + 2\sqrt{3}}} = 4.753 \quad \ldots\ldots \ (15)$$

That is, when the orbit in the velocity space is a catenary, the duration of a loop reaches the minimum:

$$T_{\min} = \sqrt{\frac{108a}{(\operatorname{arsh}\sqrt{3} + 2\sqrt{3})\mu g}} = 4.753\sqrt{\frac{a}{\mu g}} \quad \ldots\ldots \ (16)$$