Physics Cup 2022 Problem 2

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Contents

1 Three preliminary observations

1. *The boy's acceleration has constant magnitude μg.*

   We show that the boy's acceleration has a constant magnitude of μg in order to minimize the time $\mathcal{T}$ he takes to complete a full lap.

   Clearly, the boy's maximum acceleration is $\mu m_{\text{boy}} g / m_{\text{boy}} = \mu g$.

   Now, if there is a part p of the boy's trajectory where the boy's acceleration doesn't exeed $\mu g - \varepsilon$ for $\varepsilon > 0$, then he can decrease $\mathcal{T}$ by doing the following. He follows his original trajectory, but he increases his tangential acceleration by a small amount during the first half of p and decreases his tangential acceleration by a small amount during the second half. If the change in acceleration is sufficiently small, then his total acceleration will still be below μg on p. Also, with the right change in tangential acceleration on p, the change in his speed will be 0 at the start of p, increase to a maximum δv_{m} somewhere in the middle of p, and decrease back to 0 at the end of p. (The change in speed must be 0 at the two ends of p to ensure that speed is continuous at these two points.) So his average speed increased by a nonzero amount on p, meaning it now takes him a shorter time to traverse p and thus a shorter time to complete the whole lap.

   Some parts in the argument are somewhat hand-wavy. **Filling in the details is left as an exercise for the reader :P**

2. *There are no constraints on the direction of the boy's acceleration.*

   It is well known that an upright person can exert static friction on the ground in any direction and of any magnitude (up to the maximum allowed by the coefficient friction) at his will by precisely controlling his body movements. Thus, the boy is capable of accelerating at μg in any direction.

3. *The boy's trajectory passes through all vertices of the triangle.*

   We show that, the boy's trajectory must pass through all vertices of the triangle when $\mathcal{T}$ is minimized.

   First, note that the minimum period $\mathcal{T}_{\text{m}}(a)$ as a function of a is strictly increasing. In particular, since $\mathcal{T}_{\text{m}}$ is only dependent on the size of the triangle a and the boy's acceleration μg, dimensional analysis gives $\mathcal{T}_{\text{m}} \propto \sqrt{\frac{a}{\mu g}}$, which strictly increases with increasing a.

   Now, suppose that the boy's period-minimizing trajectory doesn't pass through all vertices of the triangle. Then a strictly larger triangle of side length $a' > a$ fits within the boy's trajectory of period $\mathcal{T}_{\text{m}}(a)$, meaning that $\mathcal{T}_{\text{m}}(a) \geq \mathcal{T}_{\text{m}}(a')$, which contradicts the fact that $\mathcal{T}_{\text{m}}(a)$ is a strictly increasing function.

The three observations above indicate that the original problem is equivalent to the following:

> A boy (modeled as a point mass) undergoes periodic motion in the plane around an equilateral triangle of side length a. His trajectory passes through all three vertices of the triangle. Also, his acceleration has constant magnitude μg but there are no constraints on its direction. Find the minimum period $\mathcal{T}$ of the boy's motion.

The rest of this article focuses on the equivalent problem stated above.

Here are some preliminary definitions:

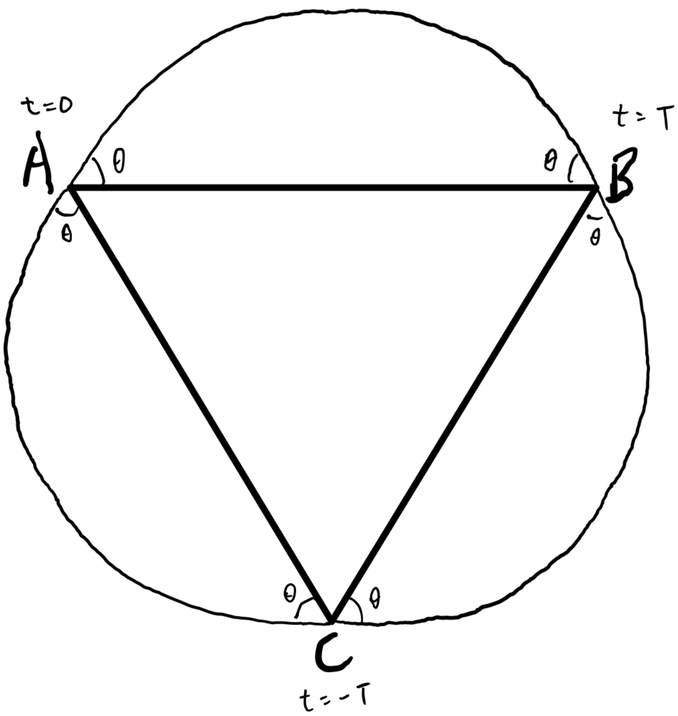
- Let $\mathbf{r}(t)$ be the trajectory of the boy, which is periodic with period $\mathcal{T}$. WLOG, we assume the boy runs clockwise around the triangle.
- Let the equilateral triangle be ABC (vertices labeled in clockwise order) and let its center be O. Let $\mathbf{r}(0) = A$.

2 Symmetry assumptions

Due to the symmetry of the equilateral triangle, we assume the period-minimizing trajectory is both rotationally symmetric and reflectionally symmetric. To be precise, we're assuming 3-fold rotational symmetry, which says that the trajectory on each side of the triangle is the same: the point $\mathbf{r}(t)$ coincides with $\mathbf{r}(t-T)$ rotated clockwise by $2\pi/3$, where $T = \mathcal{T}/3$. In addition, we assume reflectional symmetry across each axis of symmetry of the triangle: the point $\mathbf{r}(t)$ coincides with $\mathbf{r}(-t)$ reflected across OA.

Due to rotational symmetry, the boy's trajectory is the same on the three sides of the triangle, so minimizing the total period $\mathcal{T}$ is equivalent to minimizing the time $T = \mathcal{T}/3$ it takes to go from one vertex of the triangle to the next.

Due to reflectional symmetry, the portion of the trajectory between A and B (i.e., $p_{AB} := \mathbf{r}([0,T])$) makes the same angle θ with AB at A and B. And due to rotational symmetry this angle is the same for $p_{BC} := \mathbf{r}([T, 2T])$ and $p_{CA} := \mathbf{r}([-T, 0])$. Also due to symmetry, the boy's speeds right before and right after passing through each vertex of the triangle (there are 6 such speeds) are all equal, and call that speed v_0. There are two cases that we will discuss separately: $\theta \neq \pi/3$ (Section 3) and $\theta = \pi/3$ (Section 4).



A rigorous justification that the period-minimizing trajectory satisfies rotational and reflectional symmetry is given in the Appendix (Section 5).

3 Case 1: $\theta \neq \frac{\pi}{3}$

Here, p_{CA} and p_{AB} meet at an angle $2\theta + \frac{\pi}{3} \neq \pi$ at A, meaning they form a corner at A. Since acceleration is always finite, the boy's speed v_0 at A must be zero. We show that in this case that

$$T \geq 2\sqrt{\frac{a}{\mu g}}.$$

Consider just p_{AB}. The length is at least a, and the magnitude of the boy's tangential acceleration is at most μg everywhere.

Within a fixed time T, the maximum distance the boy can travel while keeping the magnitude of his tangential acceleration $\frac{dv}{dt}$ (v is speed) at most μg is

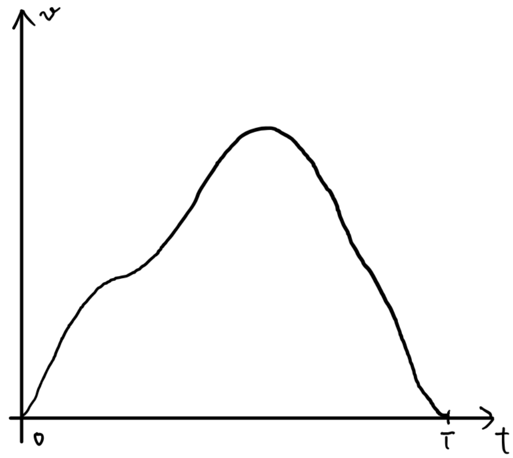
$$d_{\mathrm{m}} = \frac{1}{4}\mu g T^2.$$

We can see this by considering the v-t graph of the boy. The graph starts at $(0, 0)$ and ends at $(T, 0)$ because the boy's speed is zero at A and B, and the slope at any point on the graph is always between $\pm\mu g$. The distance the boy travles is given by the area under the graph, which is maximized when the graph goes consists of the line from $(0, 0)$ to $\left(\frac{T}{2}, \mu g \frac{T}{2}\right)$ (slope μg) and the line from $\left(\frac{T}{2}, \mu g \frac{T}{2}\right)$ to $(T, 0)$ (slope $-\mu g$). That gives a maximum distance

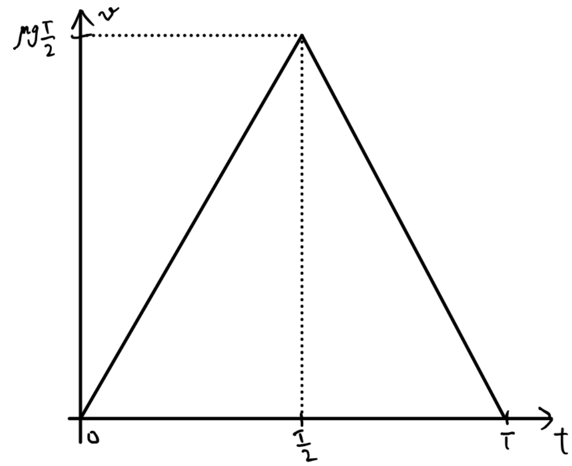
$$d_{\mathrm{m}} = \frac{1}{2}T\left(\mu g \frac{T}{2}\right) = \frac{1}{4}\mu g T^2.$$

Now, this distance must be at least a, i.e.,

$$\begin{aligned} \frac{1}{4}\mu g T^2 &\geq a \\ T &\geq 2\sqrt{\frac{a}{\mu g}}. \end{aligned}$$



v
0
T
t

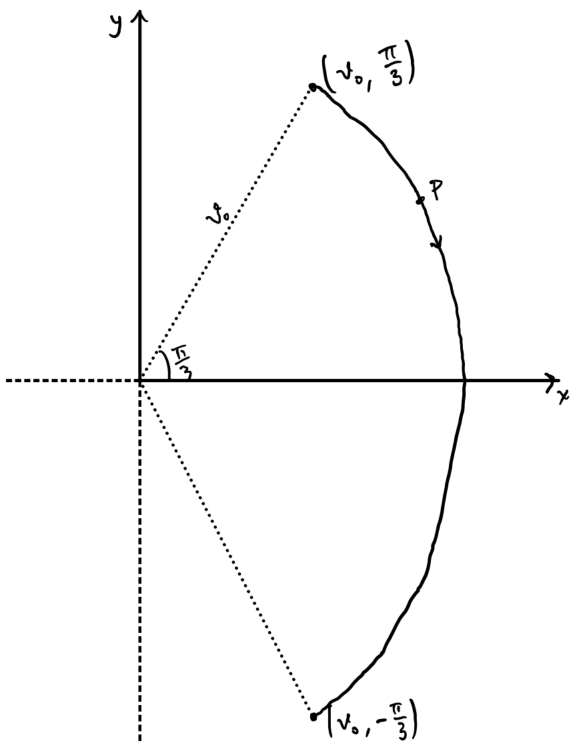


v
$\mu g \frac{T}{2}$
0
$\frac{T}{2}$
T
t

4 Case 2: $\theta = \frac{\pi}{3}$

Now p_{CA} and p_{AB} meet at an angle $2\theta + \frac{\pi}{3} = \pi$, so the boy's speed v_0 at A can be nonzero. In this case, the boy's velocity is continuous at A as both its magnitude and direction are continuous at A, so the acceleration at A would simply be the limit approaching A from either side, i.e., μg.

4.1 Equivalent problem in velocity space



Now, consider the boy's trajectory p_{AB} in velocity space. We denote the boy's velocity $\mathbf{v}(t)$ by point P in velocity space. Recall that $|\mathbf{v}(0)| = |\mathbf{v}(T)| = v_0$. Now, conditions in position space can be translated to conditions in velocity space as follows.

In polar coordinates, P starts at $\mathbf{v}(0) = (v_0, \theta)$ at time $t = 0$ and ends at $\mathbf{v}(T) = (v_0, -\theta)$ at time $t = T$, where $\theta = \frac{\pi}{3}$.

- "The angle θ that p_{AB} makes with AB at points A and B equals $\frac{\pi}{3}$" becomes "$\mathbf{v}(0)$ points in the direction of $\frac{\pi}{3}$ and $\mathbf{v}(T)$ points in the direction of $-\frac{\pi}{3}$." Thus, the polar coordinates of P at time $t = 0$ are given by $\mathbf{v}(0) = \left(v_0, \frac{\pi}{3}\right)$ and, at time $t = T$, $\mathbf{v}(T) = \left(v_0, -\frac{\pi}{3}\right)$.

- "The magnitude of the boy's acceleration is always μg" (i.e., $|\mathbf{r}''(t)| = \mu g$) becomes "the speed of P is always μg" (i.e., $|\mathbf{v}'(t)| = \mu g$).
- "The boy starts at A at $t = 0$ and ends at B at $t = T$" (i.e., $\mathbf{r}(T) - \mathbf{r}(0) = \overrightarrow{AB} = a\hat{\mathbf{i}}$) becomes "$\mathbf{X} := \int_0^T \mathbf{v}(t)\,dt = a\hat{\mathbf{i}}$."

Also, because we assumed the trajectory is symmetric (i.e., $\mathbf{r}(t)$ is $\mathbf{r}(T-t)$ reflected across the vertical), the trajectory of P is also symmetric: $\mathbf{r}'(t) = \mathbf{v}(t)$ is $\mathbf{r}'(T-t) = -\mathbf{v}(T-t)$ reflected across the vertical, or in other words, $\mathbf{v}(t)$ is $\mathbf{v}(T-t)$ reflected across the horizontal.

4.2 Minimizing T with a fixed is equivalent to maximizing a with T fixed

We now wish to minimize T subject to the three conditions stated above while fixing a and μg. We show that this is equivalent to maximizing a while fixing T and μg. As we have shown before, the minimum period T_m is a strictly-increasing function $f(a)$ of the size a of the triangle (see third observation in Section 1). Now, the feasible (T, a) for a fixed μg are described by the inequality $T \geq f(a)$, which is equivalent to $a \leq f^{-1}(T)$ because f is strictly increasing. So the problem of finding T_{m} given a and μg is equivalent to finding a_{m}, i.e., the maximum a given T and μg. Hence, from now on, we fix T and μg and maximize a subject to the three conditions listed.

4.3 Equivalent problem of minimizing the potential energy of a rope in velocity space

Now that we've fixed T, note that P traces out a path p of fixed length $L = \mu g T$ in velocity space, where μg is P's speed. In addition, recalling the symmetry of P's trajectory, $\mathbf{X} := \int_0^T \mathbf{v}(t)\,dt$ points in the x-direction, so maximizing a where $\mathbf{X} = a\hat{\mathbf{i}}$ is equivalent to maximizing $a = \mathbf{X} \cdot \hat{\mathbf{i}}$.

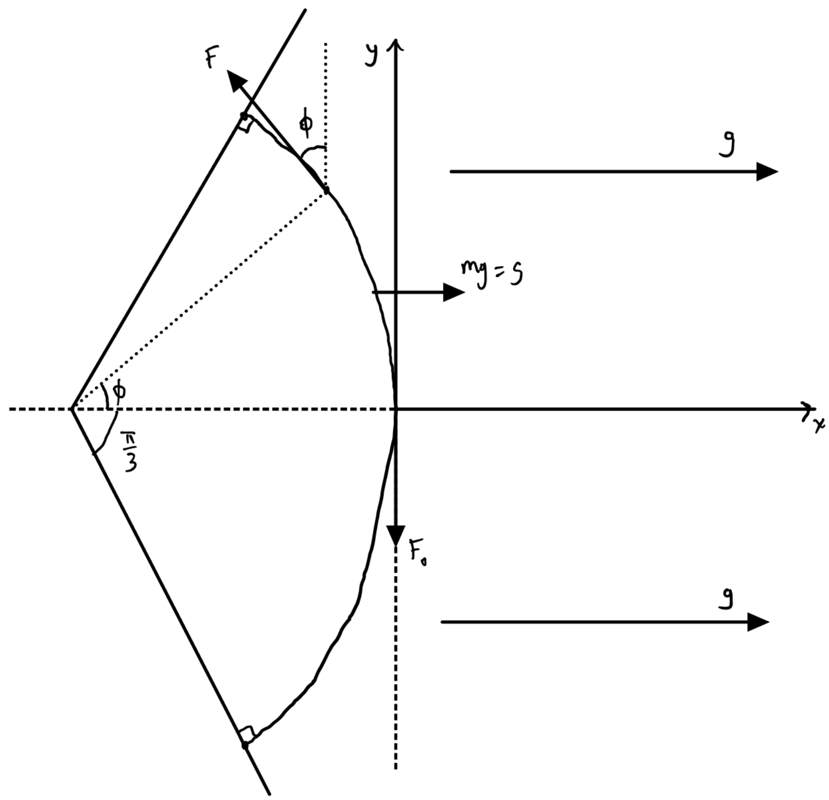
On the other hand, imagine p as a rope of uniform unit linear density. Its length is given ($L = \mu g T$) and its two ends are at $\left(v_0, \frac{\pi}{3}\right)$ and $\left(v_0, -\frac{\pi}{3}\right)$ where v_0 can be chosen freely. Maximizing $\mathbf{X} \cdot \hat{\mathbf{i}}$ is then equivalent to maximizing

$$\mu g \mathbf{X} \cdot \hat{\mathbf{i}} = \int_p \tilde{\mathbf{v}}(s)\,ds \cdot \hat{\mathbf{i}}, \tag{1}$$

where $ds = \mu g\,dt$ is the length of a small piece of the rope and $\tilde{\mathbf{v}}(s) = \mathbf{v}(t)$ is p parameterized by arc length. But recall that a differential element of the rope between s and $s + ds$ has mass ds and position vector $\tilde{\mathbf{v}}(s)$, so when subjected to a unit uniform gravitational field $\hat{\mathbf{i}}$, the rope element's gravitational potential energy is $-(ds)\tilde{\mathbf{v}}(s) \cdot \hat{\mathbf{i}}$. In other words, the RHS of (1) is just negative the potential energy of the rope. So to maximize (1), we minimize the potential energy of the rope.

When the potential energy of the rope is minimized, it's in equilibrium, and we can deduce the equilibrium configuration of the rope using statics.

4.4 Solving the equivalent problem of a rope in velocity space



We have a uniform rope of length $L = \mu g T$ and unit linear density with each end attached to a long rod. One rod is a ray from the origin O pointing in the direction of $\pi/3$, and the other rod is a ray from O pointing in the direction of $-\pi/3$. The ends of the rope are free to slide frictionlessly along the rods, and the whole system is in a unit uniform gravitational field $\hat{\mathbf{i}}$. We wish to find the configuration of the rope at minimum potential energy, where the rope will be in equilibrium. For now, we ignore the constraint that the two ends of the rope must be the same distance v_0 from O: it will turn out that the equilibrium configuration of the rope automatically satisfies that constraint.

A differential element at one end of the rope receives a normal force from the rod and tension from the rope, which must cancel. Therefore, tension is perpendicular to the rod at each end of the rope, meaning the rope makes $90°$-angles with the rods.

The rope takes on the shape of a catenary, the equation of which we will now derive.

We will work in the coordinate system $\tilde{x}\tilde{O}\tilde{y}$ where the origin $\tilde{O}$ is at the point on the rope where the tangent is vertical. Now, consider a point $P(\tilde{x}, \tilde{y})$ on the rope, and consider the rope segment between $\tilde{O}$ and P. It experiences three forces:

- Downward tension F_0 at the bottom end;
- Tension F at the top end, at an angle ϕ from the vertical;
- Gravity $mg = s$ to the right, where $s = \int_0^{\tilde{y}} \sqrt{1 + \left(\frac{d\tilde{x}}{d\tilde{y}}\right)^2}\, d\tilde{y}$ is the length of the segment.

Force balance in the vertical direction gives $F \cos\phi = F_0$, and force balance in the horizontal direction gives

$$
\begin{aligned}
s &= F \sin\phi = F_0 \tan\phi \\
s^2 &= F_0^2 \left(\sec^2\phi - 1\right) \\
s^2 &= F_0^2 \left(\left(\frac{ds}{d\tilde{y}}\right)^2 - 1\right) \\
\frac{ds}{d\tilde{y}} &= \sqrt{\left(\frac{s}{F_0}\right)^2 + 1} \\
\int_0^s \frac{ds}{\sqrt{(s/F_0)^2 + 1}} &= \int_0^{\tilde{y}} d\tilde{y} \\
F_0 \sinh^{-1}\left(\frac{s}{F_0}\right) &= \tilde{y} \\
s &= F_0 \sinh\left(\frac{\tilde{y}}{F_0}\right) \\
\tilde{x} &= -\int_0^{\tilde{y}} \sqrt{\left(\frac{ds}{d\tilde{y}}\right)^2 - 1}\, d\tilde{y} \\
&= -\int_0^{\tilde{y}} \sqrt{\cosh^2\left(\frac{\tilde{y}}{F_0}\right) - 1}\, d\tilde{y} \\
&= -\int_0^{\tilde{y}} \sinh\left(\frac{\tilde{y}}{F_0}\right) d\tilde{y} \\
\tilde{x} &= -F_0 \cosh\left(\frac{\tilde{y}}{F_0}\right).
\end{aligned}
$$

Now, at each end of the rope, the rope is perpendicular to the rod at that end, meaning

$\phi = \pm\pi/3$ ($\pi/3$ at the top end and $-\pi/3$ at the bottom end). In other words,

$$\begin{aligned}
-\frac{d\tilde{x}}{d\tilde{y}} &= \tan\phi = \tan\left(\pm\frac{\pi}{3}\right) \\
\sinh\left(\frac{\tilde{y}}{F_0}\right) &= \pm\sqrt{3} \\
\tilde{y} &= \pm F_0\alpha,
\end{aligned}$$

where

$$\alpha := \sinh^{-1}\sqrt{3} = \cosh^{-1} 2 = \ln\left(2+\sqrt{3}\right).$$

So, the total length of the rope is

$$\begin{aligned}
L &= s|_{\tilde{y}=F_0\alpha} - s|_{\tilde{y}=-F_0\alpha} \\
&= 2F_0 \sinh(\alpha) \\
&= 2\sqrt{3}F_0 \\
F_0 &= \frac{L}{2\sqrt{3}}.
\end{aligned}$$

Now, at the two ends of the rope, $\tilde{y} = \pm F_0\alpha$ and $\tilde{x} = -F_0\cosh\left(\frac{\tilde{y}}{F_0}\right) = -F_0\cosh\alpha = -2F_0$. Therefore, while O has $\tilde{y}$-coordinate 0 by symmetry, its $\tilde{x}$-coordinate is

$$-2F_0 - \frac{F_0\alpha}{\sqrt{3}} = -\left(2+\frac{\alpha}{\sqrt{3}}\right)F_0.$$

Therefore, to transform back to the coordinate system where O is the origin, we use

$$\begin{aligned}
x &= \tilde{x} + \left(2+\frac{\alpha}{\sqrt{3}}\right)F_0 \\
y &= \tilde{y}.
\end{aligned}$$

So, the equation of the rope is given by

$$x = \left(2 + \frac{\alpha}{\sqrt{3}} - \cosh\left(\frac{y}{F_0}\right)\right)F_0, \tag{2}$$

where $F_0 = L/2\sqrt{3}$.

4.5 Finishing up

Equation (2) gives the rope's configuration at minimum potential energy, where a is maximized to

$$
\begin{aligned}
a_{\mathrm{m}} &= \mathbf{X}\cdot\hat{\mathbf{i}} \\
&= \frac{1}{\mu g}\int_p \tilde{\mathbf{v}}(s)\,ds\cdot\hat{\mathbf{i}} && \text{(recall (1))} \\
&= \frac{1}{\mu g}\int_p x\,ds \\
&= \frac{1}{\mu g}\int_{-F_0\alpha}^{F_0\alpha} x\frac{ds}{dy}\,dy \\
&= \frac{1}{\mu g}\int_{-F_0\alpha}^{F_0\alpha}\left(2+\frac{\alpha}{\sqrt{3}}-\cosh\left(\frac{y}{F_0}\right)\right)F_0\cosh\left(\frac{y}{F_0}\right)\,dy \\
&= \frac{F_0^2}{\mu g}\int_{-\alpha}^{\alpha}\left(2+\frac{\alpha}{\sqrt{3}}-\cosh u\right)\cosh u\,du && (y=F_0u) \\
&= \frac{L^2}{12\mu g}\int_{-\alpha}^{\alpha}\left[\left(2+\frac{\alpha}{\sqrt{3}}\right)\cosh u-\cosh^2 u\right]du \\
&= \frac{1}{12}\mu g T^2\left[\left(2+\frac{\alpha}{\sqrt{3}}\right)\sinh u-\frac{1}{2}\sinh u\cosh u-\frac{1}{2}u\right]_{-\alpha}^{\alpha} \\
&= \frac{1}{6}\mu g T^2\left(\left(2+\frac{\alpha}{\sqrt{3}}\right)\sinh\alpha-\frac{1}{2}\sinh\alpha\cosh\alpha-\frac{1}{2}\alpha\right) \\
&= \frac{1}{6}\mu g T^2\left(2\sqrt{3}+\alpha-\sqrt{3}-\frac{1}{2}\alpha\right) \\
&= \frac{1}{12}\mu g T^2\left(2\sqrt{3}+\alpha\right).
\end{aligned}
$$

(We've used the fact that $\int\cosh^2 u\,du = \frac{1}{2}\int(\cosh 2u+1)\,du = \frac{1}{4}\sinh 2u+\frac{1}{2}u+C = \frac{1}{2}\sinh u\cosh u+\frac{1}{2}u+C$.)

Therefore, the minimum T at a given a satisfies

$$
\begin{aligned}
a &= \frac{1}{12}\mu g T_{\mathrm{m}}^2\left(2\sqrt{3}+\alpha\right) \\
T_{\mathrm{m}} &= 2\sqrt{\frac{3}{2\sqrt{3}+\alpha}\frac{a}{\mu g}}\approx 1.584\sqrt{\frac{a}{\mu g}}. && (3)
\end{aligned}
$$

This result is less than the $2\sqrt{\frac{a}{\mu g}}$ we got in Case 1 (Section 3), so (3) indeed gives the minimum T over all possible trajectories of the boy.

To conclude, the minimum period of the boy is

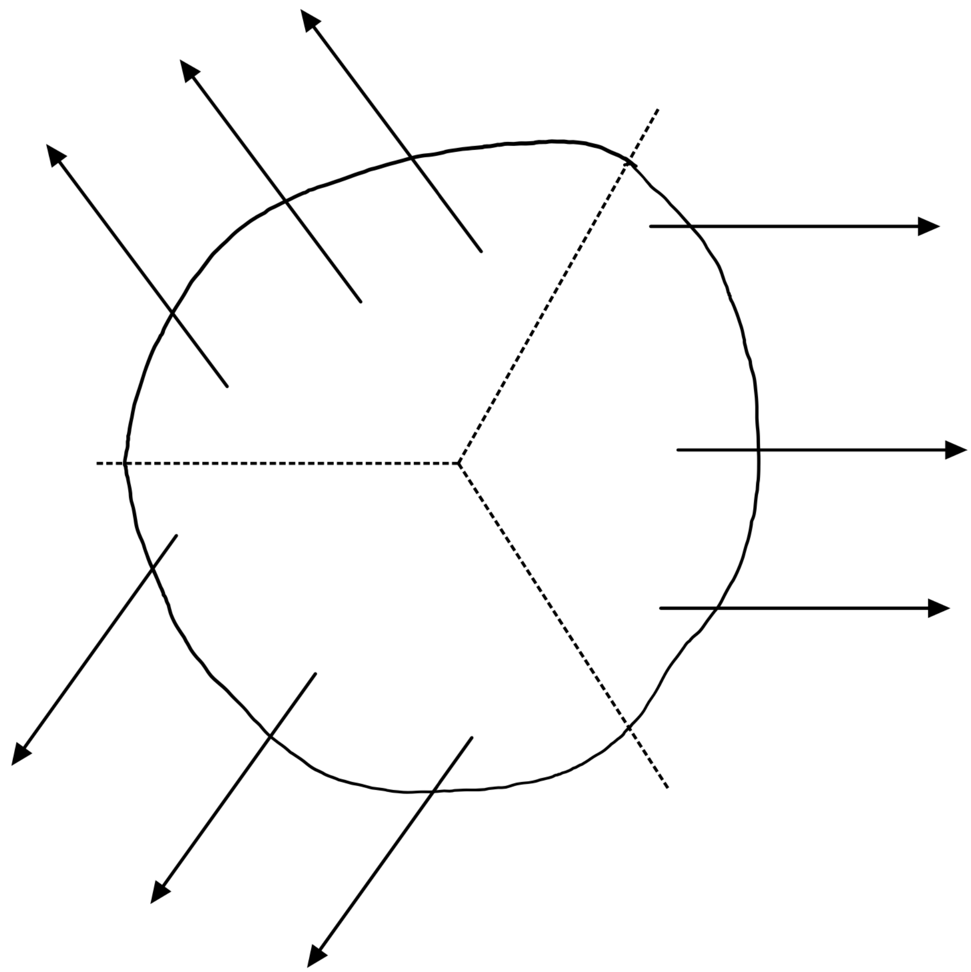
$$\mathcal{T}_{\mathrm{m}} = 3T_{\mathrm{m}} = 6\sqrt{\frac{3}{2\sqrt{3}+\alpha}\frac{a}{\mu g}},$$

where

$$\alpha = \ln\left(2+\sqrt{3}\right).$$

5 Appendix: Proof of symmetry for the period-minimizing trajectory

We later discovered that the proof here is wrong. But I think it would be cool to put the wrong proof here anyways. Figuring out why it is wrong is left as an exercise for the reader :P



Given any valid trajectory in physical space (one that has constant acceleration magnitude and passes through the vertices of the house), we can consider the corresponding trajectory in velocity space, which can be thought of as a closed rope of unit uniform linear density. Divide the velocity trajectory into 3 sections, one for each side of the triangle the boy is on. Let there be a unit uniform force field acting on each section in the direction parallel to the side of the house it pertains to. A valid trajectory in velocity space must satisfy the following constraints:

1. Consider one section (WLOG let it be the section corresponding to the AB). The potential energy of the rope is equal to negative the side length of the house $-\mu ga$ (recall (1)). (We define the origin to be at zero potential.)

2. Consider the same section. The integral of the y-component over arc length is 0.

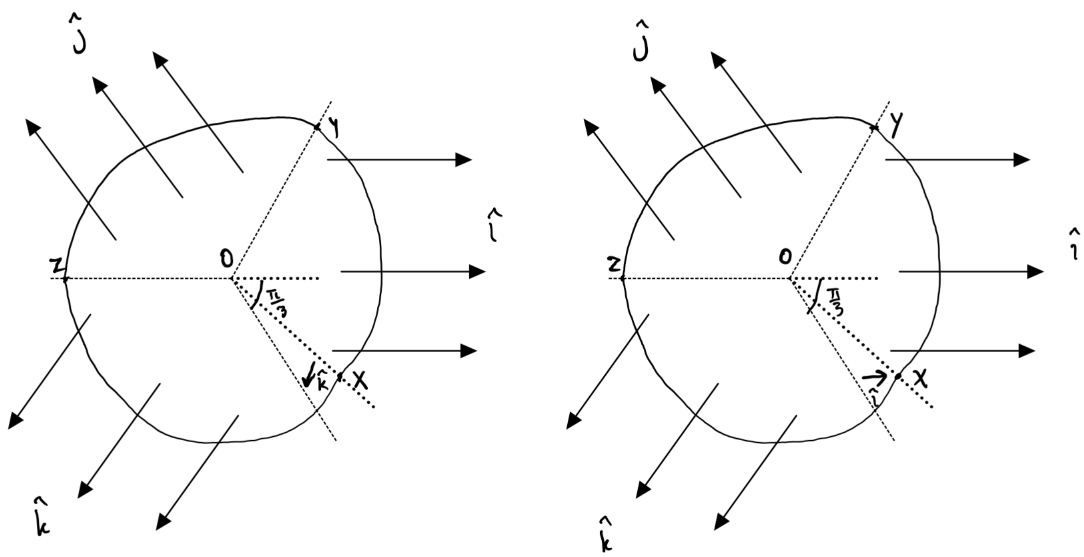
3. The total potential energy of the system is equal to $-3\mu ga$. (Note that this is strictly weaker than constraint 1)

4. The entire trajectory in velocity space must be continuous.

The goal is to minimize the total length of the rope $L = \mu g\mathcal{T}$. Let's ignore the first 2 constraints and solve the resulting problem. It will turn out that the solution will automatically satisfy the first 2 constraints.

As shown before, minimizing L with a given is equivalent to maximizing a with L given. But that's just equivalent to minimizing the total potential energy $-3\mu ga$ of the system.

When potential energy is minimized, the system is in static equilibrium. Therefore, the three forces on the three sections of the rope cancel. The three force vectors form an equilateral triangle, meaning that they all have the same magnitude. But the force on each section of the rope has magnitude equal to its length, which implies that the three sections of the rope have equal length.



Second, let the three sections of the rope have endpoints X, Y, Z (so the ropes are XY, YZ, and ZX). Let XY, YZ, ZX be in unit uniform gravitational fields $\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}}$, where $\hat{\mathbf{\imath}}$ is in $\angle XOY$, etc. Then we show that $\angle(\overrightarrow{OX}, \hat{\mathbf{\imath}}) = \angle(\overrightarrow{OY}, \hat{\mathbf{\imath}}) = \pi/3$, and similarly for the fields $\hat{\mathbf{\jmath}}$ and $\hat{\mathbf{k}}$. Suppose not. Then, WLOG, let $\angle(\overrightarrow{OX}, \hat{\mathbf{\imath}}) < \pi/3$ and $\angle(\overrightarrow{OX}, \hat{\mathbf{k}}) > \pi/3$. Then $\overrightarrow{OX} \cdot \hat{\mathbf{\imath}} > \overrightarrow{OX} \cdot \hat{\mathbf{k}}$. Thus, if we consider the region within the angle $\angle XOP$, where P is a point in the field $\hat{\mathbf{k}}$ close to the border OX between the fields $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{k}}$, we can change the field in that region from $\hat{\mathbf{k}}$ to $\hat{\mathbf{\imath}}$. Then the potential energy of the tiny rope segment inside that region will decrease because of $\overrightarrow{OX} \cdot \hat{\mathbf{\imath}} > \overrightarrow{OX} \cdot \hat{\mathbf{k}}$. Hence, the original configuration was not at its minimum potential energy.The potential energy of the rest of the rope hasn't changed, so the potential energy of the entire rope decreased. We can then shrink the entire rope again slightly to raise the potential energy of the rope back to $-3\mu g a$, but the total length of the rope has then been reduced. Therefore, when potential energy is minimized, $\angle(\overrightarrow{OX}, \hat{\mathbf{\imath}}) = \angle(\overrightarrow{OY}, \hat{\mathbf{\imath}}) = \pi/3$, and similarly for ropes YZ and ZX.

Thus, it remains to minimize the potential energy of each rope segment (which has the same length, as we have shown), where the potential field $\hat{\mathbf{\imath}}$ is restricted in the region $\angle XOY$ where $\angle(\overrightarrow{OX}, \hat{\mathbf{\imath}}) = \angle(\overrightarrow{OY}, \hat{\mathbf{\imath}}) = \pi/3$, and similarly for the fields $\hat{\mathbf{\jmath}}$ and $\hat{\mathbf{k}}$. This was done in Section 4.