The outline of the solution is as follows. Firstly, I will make a few preliminary remarks regarding the symmetry, shape and size of the trajectory. Then, I will use methods from the calculus of variations to find under what conditions the time is minimal. Finally, I will numerically calculate the minimal time needed for a complete loop.

Summary

1 Initial considerations

- The only length scale in the problem is the side length of the house, a, and the only acceleration scale is the maximal acceleration due to friction, μg (since the vertical gravitational acceleration itself doesn't directly affect the motion in the horizontal plane at all). Using dimensional analysis, it can immediately be seen that the expression for the minimal time, T_m, must be of the form

$$T_m = C\sqrt{\frac{a}{\mu g}} \equiv C\tau, \tag{1}$$

where C is a constant that will, in what follows, be found.

- According to (1), the time decreases for decreasing a. This means that the optimal trajectory is as tight as possible around the house. If the trajectory would not touch any one of the vertices of the triangle, then it would always be possible to shrink it (and eventually translate it) so that it keeps the same shape, up to homothety. And such a shrinkage would invariably lead to a decrease in the time needed. So, **the optimal trajectory is one that touches all three of the vertices of the triangle**.

- If the curve defining the trajectory would contains *kinks* (points where the first derivative of the function is not continuous), then their presence would require that the boy have a speed of 0 in that point (because the first derivative of the speed of the boy is always continuous). This is almost certainly not to be wanted (because there a would be a significant time excess needed for the boy to stop, then start moving again). So, **I will search for trajectories that do not contain kinks**.

- At any time, the frictional force can be used either to increase the speed of the boy or to tighten its trajectory. Both modes of usage lead to a shortening of the time. And the frictional force can be used at all times in this scope. Hence, the optimal trajectory is one on which **the absolute value of the acceleration due to the frictional force is always maximal, that is, it is μg**.

- If the trajectory were not rotationally symmetric, then there would be multiple trajectories that lead to the same minimal time (one could simply rotate a trajectory by $2\pi/3$ or $4\pi/3$). This is unlikely, and so a reasonable assumption is that the trajectory has rotational symmetry of order 3. Furthermore, the same would occur if the trajectory would not have mirror symmetry with respect to any axis of symmetry of the triangle. Hence, I will also assume that the trajectory has such symmetry. In what follows, I will group these assumptions under the name of **the assumption of symmetry**.

- Under the assumption of symmetry, let T_{side} be the time needed to move from a vertex to the next. Then, $T = 3T_{side}$ (where T is the time needed to make a full loop) and the problem reduces to that of finding the minimal T_{side} (under certain conditions, discussed below).
- Assuming that the trajectory is symmetric, the angle between the velocity vector and the radial direction must be the same on both sides of a vertex of the triangle. Coupled with the assumption of no kinks, this implies that the velocity vector in such a point must be tangential, which further means that **the angle between the velocity in a vertex and the next side of the triangle is** $60°$.
- Several other consequences of the symmetry of the trajectory are:
  - The speed of the boy in each of the vertices is the same; let this speed be called v_0.
  - The trajectory is also symmetric with respect to the perpendicular bisector of a side. Using the notations in the figure below, this means that the velocity of the boy in point M is tangential.
  - For the same reasons, the time it takes to get from A to M is the same as the time it takes to get from M to B. Let T_0 be this time. Then, according to what was said above, $T_m = 6T_0$, and the problem reduces to minimizing T_0, under the conditions stated above.

Let us sum up what was said above. Under the assumption of symmetry, we have reduced the problem to that of minimizing T_0, under the following conditions:

- The velocity $\mathbf{v_A}$ is at an angle of $60°$ from the side AB of the triangle.
- Point M is situated on the perpendicular bisector, *i.e.* $x_M = a/2$.
- The y-component of the velocity in M is 0.
- The acceleration of the boy is always μg in magnitude.

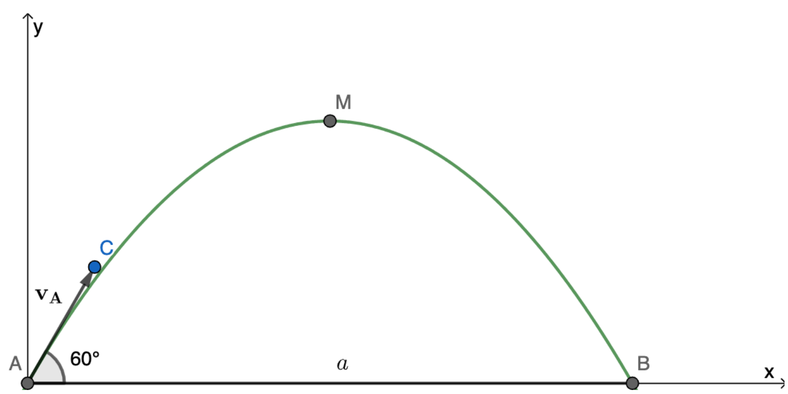

Figure 1: An example trajectory of the boy from A to B

2 Minimization of the time

2.1 Expressing the integrals I_1 and I_2

Note: From now on, the solution only considers the motion of the boy from A to M.

The direction of the acceleration can be any; its magnitude is constant. To describe the time evolution of the vector **a**, I will define $\theta(t)$ to be the angle between the acceleration and the *negative* y-axis, with counter-clockwise taken to be positive[1]. The time t is taken to be 0 when the boy is in A; it will be T_0 when he reaches M.

Using this notation, the equations of motion of the boy are

$$\begin{aligned}\dot{v_y} &= -\mu g \cos\theta;\\ \dot{v_x} &= \mu g \sin\theta.\end{aligned} \tag{2}$$

Integrating the first of these from $t = 0$ to $t = T_0$ we get

$$v_{y,M} - v_{y,A} = -\mu g \int_0^{T_0} \cos\theta dt. \tag{3}$$

But $v_{y,A} = v_0 \sin 60° = \frac{\sqrt{3}}{2} v_0$, and $v_{y,M} = 0$, so that

$$\boxed{I_1 \equiv \int_0^{T_0} \cos\theta dt = \frac{\sqrt{3}}{2}\frac{v_0}{\mu g}.} \tag{4}$$

Noting that $v_{x,A} = v_0 \cos 60° = \frac{1}{2} v_0$, we can integrate the second equation from (2) twice to obtain

$$x_M - x_A = \frac{1}{2} v_0 T_0 + \mu g \int_0^{T_0} \int_0^t \sin\theta dt' dt, \tag{5}$$

where the inner integral (whose variable is t') goes from 0 to t, and the outer one (whose variable is t), from 0 to T_0. In shorthand notation, I will write this as

$$x_M - x_A = \frac{1}{2} v_0 T_0 + \mu g \iint_0^{T_0} \sin\theta dt^2. \tag{6}$$

Since $x_A = 0$ and $x_M = a/2$,

$$\boxed{I_2 \equiv \iint_0^{T_0} \sin\theta dt^2 = \frac{a}{2\mu g} - \frac{1}{2}\frac{v_0 T_0}{\mu g}.} \tag{7}$$

We have thus obtained expressions for two integrals - I_1 and I_2 - that are related to the function $\theta(t)$ in terms of a, μ, g, v_0 and T_0.

2.2 Maximizing I_2

Considering particular, known values of v_0 and T_0, Eq. (4) tells us something about the function $\theta(t)$. Eq. (7), on the other hand, also depends on a. So by considering a to be a "variable" (which is not true, but is good enough for the present purposes), we see that we can maximize it by maximizing I_2. And we know that, if we maximize a for a certain value of T, we minimise T for that value of a, due to the dimensional arguments in Section 1. Hence, we need to maximize I_2, considering Eq. (4) known and true.

If $\theta(t)$ is a function that maximizes I_2, then I_2 must not have any first order change under small changes in $\theta(t)$, *i.e.* functions of the form $\theta(t) + \epsilon(t)$, with infinitesimal ϵ.

However, according to Eq. (4) and considering that v_0 is fixed, any such change must keep I_1 unchanged. This means that

$$I_1(\theta(t) + \epsilon(t)) = I_1(\theta(t)) \Longrightarrow \int_0^{T_0} \cos(\theta(t) + \epsilon(t)) dt = \int_0^{T_0} \cos\theta(t) dt. \tag{8}$$

[1] By this, I mean that the angle is positive if, starting from the negative y-axis, one has to revolve counter-clockwise to reach the direction of the acceleration.

But $\cos(\theta(t)+\epsilon(t)) = \cos\theta(t) - \epsilon(t)\sin\theta(t)$, so that, reducing like terms, we get

$$\boxed{\int_0^{T_0} \epsilon(t)\sin\theta(t)dt = 0.} \tag{9}$$

Similarly, the condition on the invariance of I_2 is equivalent to

$$\iint_0^{T_0} \epsilon(t)\cos\theta(t)dt^2 = 0. \tag{10}$$

More explicitly, this means that

$$\int_0^{T_0}\int_0^{t} \epsilon(t')\cos\theta(t')dt'dt = 0. \tag{11}$$

We see that the integrand is a function of t only through the upper limit of the inner integral. For a particular value of t', we see that the same term - $\epsilon(t')\cos\theta(t')$ - appears for all values of t greater than t' and is absent for $t < t'$. Therefore, in the resulting integration, this term will be multiplied by a factor of $T_0 - t'$, which is the time from t' until the end of the integration, so that the above equation is further equivalent to

$$\int_0^{T_0} (T_0 - t')\epsilon(t')\cos\theta(t')dt' = 0, \tag{12}$$

or, by changing the name of the variable,

$$\boxed{\int_0^{T_0} (T_0 - t)\epsilon(t)\cos\theta(t)dt = 0.} \tag{13}$$

The only way in which Eq. (13) can be true for *all* $\epsilon(t)$ satisfying Eq. (9) is if the function by which $\epsilon(t)$ is multiplied is the same (or if one is a constant multiple of the other):

$$(T_0 - t)\cos\theta(t) = A\sin\theta(t) \Longrightarrow \boxed{\tan\theta(t) = \tfrac{T_0 - t}{A},} \tag{14}$$

for some constant A. Hence, this is the form that a function $\theta(t)$ that maximizes I_2 - and hence minimizes the time - must take.

3 Calculation of the minimal time

Let us define θ_0 through

$$\theta_0 = \arctan\frac{T_0}{A}. \tag{15}$$

θ_0 is the initial value of θ. Using this notation, we see that

$$\tan\theta = \frac{T_0 - t}{T_0}\tan\theta_0. \tag{16}$$

We will have, then,

$$\cos\theta = \frac{A}{\sqrt{A^2 + (T_0 - t)^2}} \Longrightarrow I_1 = \frac{1}{2}A\ln\left(\frac{2T_0(\sqrt{A^2+T_0^2}+T_0}{A^2} + 1\right) = \frac{T_0}{2}\cot\theta_0\ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right); \tag{17}$$

and

$$\sin\theta = \frac{T_0 - t}{\sqrt{A^2+(T_0-t)^2}} \Longrightarrow \int_0^t \sin\theta(t)dt = \sqrt{A^2+T_0^2} - \sqrt{A^2+(T_0-t)^2}, \tag{18}$$

so that, by integrating again,

$$I_2 = \frac{1}{2}\left(T_0\sqrt{A^2+T_0^2} - A^2\ln\left(\frac{\sqrt{A^2+T_0^2}+T_0}{A}\right)\right) = \frac{T_0^2}{4}\left(\frac{2}{\sin\theta_0} - \cot^2\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right). \tag{19}$$

Eq. (4) now tells us that

$$\frac{T_0}{2}\cot\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right) = \frac{\sqrt{3}}{2}\frac{v_0}{\mu g}, \tag{20}$$

while Eq. (7) will mean that

$$\frac{T_0^2}{4}\left(\frac{2}{\sin\theta_0} - \cot^2\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right) = \frac{a}{2\mu g} - \frac{1}{2}\frac{v_0 T_0}{\mu g}. \tag{21}$$

Together, Eqs. (20) and (21) theoretically allow us to solve for both T_0 and θ_0 in terms of a, μ, g and v_0. However, we are not interested in the value of T_0 for a particular value of v_0 - instead, we want to know the minimum value of T_0 for any value of v_0.

Since v_0 is the only actual variable in the above system of equations, let us consider θ_0 and T_0 to be functions of v_0: $\theta_0 = \theta_0(v_0)$, $T_0 = T_0(v_0)$. Due to the above two equations, both θ_0 and T_0 are well-defined and unique for a particular value of v_0.

Now, let us search for the smallest possible value of T_0. For this value, changing v_0 by a small amount will not lead to any first-order change of T_0 - in other words, the derivative of T_0 with respect to v_0 is 0. There is no such constraint on θ_0, though; therefore, the derivative with respect to v_0 of the left side of both Eqs. (20) and (21) must be only due to the variation of θ_0. This means that

$$\frac{T_0}{2}\frac{d}{dv_0}\left(\cot\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right) = \frac{\sqrt{3}}{2}\frac{1}{\mu g} \tag{22}$$

and

$$\frac{T_0}{4}\frac{d}{dv_0}\left(\frac{2}{\sin\theta_0} - \cot^2\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right) = -\frac{1}{2}\frac{1}{\mu g}, \tag{23}$$

where we have calculated the derivative on the right side and, in the case of Eq. (23), simplified through T_0.

Dividing Eqs. (22) and (23), we find that

$$\frac{d}{dv_0}\left(\cot\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right) = -\frac{\sqrt{3}}{2}\frac{d}{dv_0}\left(\frac{2}{\sin\theta_0} - \cot^2\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right), \tag{24}$$

or, expressing the derivatives and simplifying through $\frac{d\theta_0}{dv_0}$,

$$\csc\theta_0\left(2 - \csc\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right) = -\sqrt{3}\csc\theta_0\cot\theta_0\left(-2 + \csc\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right). \tag{25}$$

Since $\csc\theta_0 \neq 0$, we get

$$2 - \csc\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right) = \sqrt{3}\cot\theta_0\left(2 - \csc\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right), \tag{26}$$

which is equivalent to

$$\left(2 - \csc\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right)(1 - \sqrt{3}\cot\theta_0) = 0. \tag{27}$$

The first factor is 0 only for $\theta_0 = 0$, which is not the case. So we are left with

$$\sqrt{3}\cot\theta_0 = 1 \Longrightarrow \theta_0 = \arctan\sqrt{3} \Longrightarrow \theta_0 = \frac{\pi}{3}. \tag{28}$$

Introducing the expression for v_0 from Eq. (20) into Eq. (21), we get

$$\frac{a}{\mu g} = \frac{T_0^2}{2}\left(\frac{2}{\sin\theta_0} - \cot^2\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right) + \frac{T_0^2}{\sqrt{3}}\cot\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right), \tag{29}$$

or

$$T_0 = \frac{1}{\sqrt{\frac{1}{\sin\theta_0} - \frac{1}{2}\cot^2\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right) + \frac{1}{\sqrt{3}}\cot\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)}}\sqrt{\frac{a}{\mu g}}. \tag{30}$$

Numerically, we find that

$$T_0 = \sqrt{\frac{6}{4\sqrt{3} + \ln(7+4\sqrt{3})}}\sqrt{\frac{a}{\mu g}} \approx 0.7921\sqrt{\frac{a}{\mu g}},$$

so that, finally,

$$T_m = 6T_0 \Longrightarrow \boxed{T_m = 4.753\sqrt{\frac{a}{\mu g}}.}$$

4 Appendix: Generalization to arbitrary regular polygons

We can generalize the above solution to the case in which the house has the shape of an arbitrary N-gon. Defining T_0 similarly, we will now have

$$T_m = 2NT_0, \tag{31}$$

and the angle between $\mathbf{v_A}$ and the corresponding side AB will be $\frac{\pi}{N}$. Using this fact, we will have

$$I_1 = \frac{v_0}{\mu g}\sin\frac{\pi}{N} \tag{32}$$

and

$$I_2 = \frac{a}{2\mu g} - \frac{v_0 T_0}{\mu g}\cos\frac{\pi}{N}. \tag{33}$$

The result of using the methods of calculus of variations shown in section 2 - Eq. (14) - is still true, as it doesn't depend on the specific form of the expressions of I_1 and I_2, and so are the expressions for I_1 and I_2 in terms of θ_0 and T_0, Eqs. (17) and (19).

Continuing along the lines of reasoning shown in section 3, we have:

$$\frac{dI_1}{dv_0} = \frac{1}{\mu g}\sin\frac{\pi}{N} \tag{34}$$

and

$$\frac{dI_2}{dv_0} = -\frac{T_0}{\mu g}\cos\frac{\pi}{N} \tag{35}$$

for the case where T_0 is minimal. Using the two above relationships, we find that

$$\frac{d}{dv_0}\left(\cot\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right) = -\tan\frac{\pi}{N}\frac{d}{dv_0}\left(\frac{1}{\sin\theta_0} - \frac{1}{2}\cot^2\theta_0 \ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right), \tag{36}$$

or, expressing the derivatives and simplifying through $\frac{d\theta_0}{dv_0}$,

$$\csc\theta_0\left(2-\csc\theta_0\ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right)=-\tan\frac{\pi}{N}\csc\theta_0\cot\theta_0\left(-2+\csc\theta_0\ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right), \tag{37}$$

from which we find that

$$\left(2-\csc\theta_0\ln\left(\frac{1+\sin\theta_0}{1-\sin\theta_0}\right)\right)\left(1-\tan\frac{\pi}{N}\cot\theta_0\right)=0. \tag{38}$$

Since $\theta_0 \neq 0$, this means that

$$1-\tan\frac{\pi}{n}\cot\theta_0=0\Longrightarrow\theta_0=\frac{\pi}{N}. \tag{39}$$

Hence, we will have

$$I_2=\frac{a}{2\mu g}-I_1T_0\cot\frac{\pi}{N}, \tag{40}$$

or

$$\frac{a}{2\mu g}=\frac{T_0^2}{2}\cot^2\frac{\pi}{N}\ln\left(\frac{1+\sin\frac{\pi}{N}}{1-\sin\frac{\pi}{N}}\right)+\frac{T_0^2}{2}\left(\frac{1}{\sin\frac{\pi}{N}}-\frac{1}{2}\cot^2\frac{\pi}{N}\ln\left(\frac{1+\sin\frac{\pi}{N}}{1-\sin\frac{\pi}{N}}\right)\right) \tag{41}$$

or

$$T_0=\frac{1}{\sqrt{\frac{1}{\sin\frac{\pi}{N}}+\frac{1}{2}\cot^2\frac{\pi}{N}\ln\left(\frac{1+\sin\frac{\pi}{N}}{1-\sin\frac{\pi}{N}}\right)}}\sqrt{\frac{a}{\mu g}}. \tag{42}$$

Hence,

$$T_m=2NT_0\Longrightarrow \boxed{T_m=\frac{2N}{\sqrt{\frac{1}{\sin\frac{\pi}{N}}+\frac{1}{2}\cot^2\frac{\pi}{N}\ln\left(\frac{1+\sin\frac{\pi}{N}}{1-\sin\frac{\pi}{N}}\right)}}\sqrt{\frac{a}{\mu g}}.} \tag{43}$$

N	T, in units of $\sqrt{\frac{a}{\mu g}}$
2	4.000
3	4.753
5	5.794
10	7.992
50	17.73

Table 1: A few numerical values for T_m

Note: The result $T_m = 4\sqrt{\frac{a}{\mu g}}$ in the case $N = 2$ is understood as the limit $N \to 2$. In this case, the house has the shape of a line segment, and the ideal trajectory corresponds to stopping at the ends of this segment and accelerating - and deccelerating - along the sides of this segment.

The limit $N \to \infty$

In the limit of large N, we have

$$\frac{\pi}{N} \ll 1\text{rad} \Longrightarrow \sin\frac{\pi}{N} \simeq \frac{\pi}{N}, \cot\frac{\pi}{N} \simeq \frac{N}{\pi}. \tag{44}$$

Additionally,

$$\ln\left(\frac{1+\sin\frac{\pi}{N}}{1-\sin\frac{\pi}{N}}\right) \simeq 2\frac{\pi}{N}. \tag{45}$$

Hence,

$$T_m \simeq \frac{2N}{\sqrt{\frac{N}{\pi}+\frac{1}{2}\frac{N^2}{\pi^2}2\frac{\pi}{N}}}\sqrt{\frac{a}{\mu g}} \Longrightarrow T_m \simeq \sqrt{2N\pi}\sqrt{\frac{a}{\mu g}}. \tag{46}$$

Now, let the circumradius of the N-gon pe R. Then, we will have

$$a = 2R\sin\frac{\pi}{N} \Longrightarrow a \simeq \frac{2\pi R}{N}; \tag{47}$$

plugging this into Eq. (46), we find that

$$T_m \simeq 2\pi\sqrt{\frac{R}{\mu g}}, \tag{48}$$

which is, as expected, just the time needed to make a circle of radius R.