Problem No 3 – Solution

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Relativistic units are used in the entirety of this paper.

Consider the following: An object of mass m at rest in frame S is accelerated via the emission of a single photon. The proper final mass of the object is given by κm, where $0 < \kappa < 1$.

Lemma 1. *The final energy and momentum (magnitude) of the object as observed in frame S is given by*

$$\begin{aligned} E &= \frac{1}{2}m\left(1+\kappa^2\right) \\ p &= \frac{1}{2}m\left(1-\kappa^2\right) \end{aligned} \tag{1}$$

Proof. Let $\boldsymbol{P_i}$, $\boldsymbol{P_f}$ and $\boldsymbol{P_p}$ be the **4-momenta** of the initial object, final object and photon respectively. In frame S, these take the form

$$\boldsymbol{P_i} = (m, 0, 0, 0)$$

$$\boldsymbol{P_f} = (E, p, 0, 0)$$

$$\boldsymbol{P_p} = (p, p, 0, 0)$$

where all symbols are defined as above. By the conservation of 4-momentum, we can solve for E,

$$\begin{aligned} \boldsymbol{P_p} &= \boldsymbol{P_i} - \boldsymbol{P_f} \\ \boldsymbol{P_p}^2 &= \boldsymbol{P_i}^2 - 2\boldsymbol{P_i}\cdot\boldsymbol{P_f} + \boldsymbol{P_f}^2 \\ 0 &= m^2 - 2mE + (\kappa m)^2 \\ E &= \frac{1}{2}m\left(1+\kappa^2\right) \end{aligned}$$

where we have used the fact that light-like vectors are of length zero. Using the mass-shell relation, one can easily find the associated momentum,

$$\begin{aligned} p^2 = E^2 - (\kappa m)^2 &= \frac{1}{4}m^2(1+2\kappa^2+\kappa^4) - m^2\kappa^2 \\ p &= \frac{1}{2}m\left(1-\kappa^2\right) \end{aligned}$$

which concludes the proof.

Corollary 2. *The same formulas apply if the object is accelerated via the emission of a series of photons in the same direction.*

Proof. As the series of photons emitted are in the same direction, their 4-momenta are linearly dependent (due to the vector being light-like). Thus, their vector sum is also light-like, and the above proof(s) follow.

The implication of this is that the final momentum of the rocket only depends on the mass lost during the two phases of acceleration; in other words the specific thrust profile is of no physical significance, and the total acceleration (during each acceleration phase) might as well be due to a single photon.

We now perform our analysis in the intermediate frame S, that is, the frame of the rocket just after the first acceleration phase. Letting $m_m = \eta m$ and $m_f = fm$ be the intermediate and final masses of the rocket respectively, we derive the final energy and momentum of the rocket using equation 1.

$$\begin{aligned} E_f &= \frac{m_m}{2}\left[1 + (m_f/m_m)^2\right] = \frac{m}{2}\left(\eta + f^2/\eta\right) \\ p_f &= \frac{m_m}{2}\left[1 - (m_f/m_m)^2\right] = \frac{m}{2}\left(\eta - f^2/\eta\right) \end{aligned} \tag{2}$$

To derive the initial energy and momentum of the rocket observed from S, we first calculate the corresponding *intermediate* quantities observed from the initial frame S',

$$\begin{aligned} E'_m &= \gamma m_m = \gamma \eta m = \frac{m}{2}\left(1 + \eta^2\right) \\ p'_m &= \gamma m_m v = \gamma \eta m v = \frac{m}{2}\left(1 - \eta^2\right) \end{aligned} \tag{3}$$

from which we observe that dividing by η yields the correct quantities.

$$\begin{aligned} E_i &= \gamma m = \frac{E'_m}{\eta} = \frac{m}{2}\left(1/\eta + \eta\right) \\ p_i &= -\gamma m v = -\frac{p'_m}{\eta} = \frac{m}{2}\left(1/\eta - \eta\right) \end{aligned} \tag{4}$$

Note that we have used the fact that the Lorentz factor γ is the same for a boost and its inverse, as well as the fact that the relative velocity takes on a minus sign when switching between frames.

We will now make use of the invariant nature of 4-vector dot products. Namely, we will evaluate $\boldsymbol{P_i} \cdot \boldsymbol{P_f}$ in both frame S and S'. Firstly in S, we have the following.

$$\begin{aligned} \boldsymbol{P_i} \cdot \boldsymbol{P_f} &= (E_i, -p_i, 0, 0) \cdot (E_f, p_f \cos\alpha, p_f \sin\alpha, 0) = E_i E_f + p_i p_f \cos\alpha \\ &= \frac{m^2}{4}\left[\left(\frac{1}{\eta} + \eta\right)\left(\eta + \frac{f^2}{\eta}\right) + \left(\frac{1}{\eta} - \eta\right)\left(\eta - \frac{f^2}{\eta}\right)\cos\alpha\right] \\ &= \frac{m^2}{4}\left[\left(1 + \eta^2 + f^2 + \frac{f^2}{\eta^2}\right) + \left(1 - \eta^2 + f^2 - \frac{f^2}{\eta^2}\right)\cos\alpha\right] \end{aligned} \tag{5}$$

Next in frame S', observe that $\boldsymbol{P_i} = (m, 0, 0, 0)$. Thus the dot product only depends on the temporal component of $\boldsymbol{P_f}$, which is the final energy of the rocket.

$$\begin{aligned} E'_f &= \gamma m_f = \gamma f m \\ \boldsymbol{P_i} \cdot \boldsymbol{P_f} &= m E'_f = \gamma f m^2 \end{aligned} \tag{6}$$

where γ can be calculated directly from the final speed of the rocket provided in the question.

$$\gamma = \left(1 - v^2\right)^{-1/2} = (1 - 16/25)^{-1/2} = (9/25)^{-1/2} = 5/3 \tag{7}$$

Finally equating equations 5 and 6, we can solve for $\cos\alpha$,

$$\begin{aligned} 4\gamma f &= \left(1 + \eta^2 + f^2 + \frac{f^2}{\eta^2}\right) + \left(1 - \eta^2 + f^2 - \frac{f^2}{\eta^2}\right)\cos\alpha \\ \cos\alpha &= \frac{4\gamma f - 1 - \eta^2 - f^2 - f^2/\eta^2}{1 - \eta^2 + f^2 - f^2/\eta^2} = 1 + \frac{4\gamma f - 2f^2 - 2}{1 - \eta^2 + f^2 - f^2/\eta^2} \end{aligned} \tag{8}$$

which we need to maximize in η. Close observation of equation 8 reveals that this is equivalent to minimizing $\eta^2 + f^2/\eta^2$ (in the denominator), which is a straightforward calculus exercise.

$$\begin{aligned} 0 &= \frac{d}{d\eta}\eta^2 + f^2/\eta^2 = 2\eta - 2f^2/\eta^3 \\ \eta &= \sqrt{f} \end{aligned}$$

Substituting into equation 8 we obtain

$$\cos\alpha = 1 + \frac{4\gamma f - 2f^2 - 2}{1 - 2f + f^2} = 1 + \frac{5/3 - 1/8 - 2}{1 - 1/2 + 1/16} = \frac{5}{27} \tag{9}$$

where we have used the calculated result for γ in equation 7 and $f = 1/4$ as provided in the question. Thus the final answer to the question is given by

$$\alpha_{min} = \arccos(5/27) \tag{10}$$