Physics Cup 2022 Problem 3

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Let us begin by consider the dynamics of the photon rocket without changing directions. Suppose that the rocket starts at rest with mass m and starts accelerating in the $+\hat{x}$ direction until it has mass κm.

Denote the final energy of the rocket be E and the total momentum of the photons be p (in the $-\hat{x}$ direction). By conservation of energy and momentum, we have

$$mc^2 = E + pc$$

$$0 = \frac{1}{c}\sqrt{E^2 - (\kappa mc^2)^2} - p$$

We may solve for E by eliminating p,

$$\left(mc^2 - E\right)^2 = E^2 - \left(\kappa mc^2\right)^2.$$

$$E = \frac{1+\kappa^2}{2}mc^2.$$

Now, by substituting the velocity of the rocket into the expression for E, we have

$$\frac{1}{\sqrt{1-v^2/c^2}} \cdot \kappa mc^2 = E = \frac{1+\kappa^2}{2}mc^2$$

Unsurprisingly, v/c depends only on κ. We obtain

$$v^2/c^2 = 1 - \left(\frac{2\kappa}{1+\kappa^2}\right)^2 = \frac{(1+\kappa)^2 \cdot (1-\kappa)^2}{\left(1+\kappa^2\right)^2}$$

Thus, we can say that, as a function of κ, the final velocity attained is

$$v(\kappa) = \frac{1-\kappa^2}{1+\kappa^2} \cdot c. \tag{1}$$

Returning to the original problem, suppose that the rocket burned κ_1 of its mass, it will be moving at speed v_1, relative to the initial rest frame. Call the instantaneous frame the rocket currently is in S.

In S, the rocket will turn an angle α, and then burn through another κ_2 of its mass, where $\kappa_1\kappa_2 = 1/4$. In S, the final speed of the rocket will be v_2.

Without loss of generality, let v_1 be in the direction of $+\hat{x}$ relative to the initial rest frame. By relativistic velocity addition, we know that the final velocity $\mathbf{v}_f$ with respect to the initial rest frame will be

$$v_{f,x} = \frac{v_1 + v_2\cos\alpha}{1 + v_1 v_2 \cos\alpha} \quad \text{and} \quad v_{f,y} = \sqrt{1 - v_1^2} \cdot \frac{v_2 \sin\alpha}{1 + v_1 v_2 \cos\alpha}$$

For convenience, we have dropped all factors of c. The final speed is then given by

$$\begin{aligned} v_f^2 &= \left[v_1^2 + 2v_1v_2\cos\alpha + v_2^2\cos^2\alpha + v_2^2\sin^2\alpha - v_1^2v_2^2\sin^2\alpha\right]\cdot\left(\frac{1}{1+v_1v_2\cos\alpha}\right)^2 \\ &= \left[v_1^2 + v_2^2 + 2v_1v_2\cos\alpha - v_1^2v_2^2 + v_1^2v_2^2\cos^2\alpha + 1 - 1\right]\cdot\left(\frac{1}{1+v_1v_2\cos\alpha}\right)^2 \\ &= \left[-\left(1-v_1^2\right)\cdot\left(1-v_2^2\right) + \left(1+v_1v_2\cos\alpha\right)^2\right]\cdot\left(\frac{1}{1+v_1v_2\cos\alpha}\right)^2 \end{aligned}$$

and finally simplified to

$$1 - v_f^2 = \frac{\left(1-v_1^2\right)\cdot\left(1-v_2^2\right)}{\left(1+v_1v_2\cos\alpha\right)^2}.$$

Now, we plug in the expression for $v(\kappa)$ from equation (1), to obtain

$$\begin{aligned} 1 - v_f^2 &= \left[1 - \left(\frac{1-\kappa_1^2}{1+\kappa_1^2}\right)^2\right]\cdot\left[1 - \left(\frac{1-\kappa_2^2}{1+\kappa_2^2}\right)^2\right]\cdot\left(\frac{1}{1+v_1v_2\cos\alpha}\right)^2 \\ &= \frac{4\kappa_1^2}{\left(1+\kappa_1^2\right)^2}\cdot\frac{4\kappa_2^2}{\left(1+\kappa_2^2\right)^2}\cdot\left(\frac{1}{1+v_1v_2\cos\alpha}\right)^2 \\ &= 16\kappa_1^2\kappa_2^2\cdot\left[\frac{1}{\left(1+\kappa_1^2\right)\cdot\left(1+\kappa_2^2\right) + \left(1-\kappa_1^2\right)\cdot\left(1-\kappa_1^2\right)\cdot\cos\alpha}\right]^2 \end{aligned}$$

or

$$\left[\left(1+\kappa_1^2\right)\cdot\left(1+\kappa_2^2\right) + \left(1-\kappa_1^2\right)\cdot\left(1-\kappa_1^2\right)\cdot\cos\alpha\right]^2 = \frac{16\kappa_1^2\kappa_2^2}{1-v_f^2} \tag{2}$$

Here, we have written everything in terms of $\alpha, v_f, \kappa_1\kappa_2$ and $\kappa_1^2+\kappa_2^2$. Recall that we have the constrain $\kappa_1\kappa_2 = 1/4$. Let $x = \kappa_1^2+\kappa_2^2$, and plugging in $v_f = 4/5$, we have

$$\left(\frac{17}{16} + x\right) + \left(\frac{17}{16} - x\right)\cos\alpha = \frac{5}{3}.$$

Writing $\cos\alpha$ in terms of x, this is equivalent to

$$\cos\alpha = f(x) = \frac{29/48 - x}{17/16 - x}. \tag{3}$$

Now, we have a simple rational function in terms of x. By inspection, there is a vertical asymptote at $x = 17/16$ and a horizontal asymptote towards $f(x) = 1$.

Recall that $x = \kappa_1^2+\kappa_2^2$. By the constrain $\kappa_1\kappa_2 = 1/4$, the minimum value of x is $1/2$. On the other hand, the maximum value of x is $17/16$, attained when one of κ is 1. Physically, a smaller x means that the turning point is near "half way", where half the fuel has been burnt. A larger x means that out of the two sections of the journey, one section is longer (in terms of κ) then the other.

To minimize α, we want $f(x)$ to be large while maintaining $f(x) \leq 1$. On the interval $(-\infty, 17/16)$, however, x is monotone decreasing. Thus, it is optimal to take small $x = 1/2$, or having the rocket turn "half way".

Our final answer is

$$\alpha = \cos^{-1}\left(\frac{5}{27}\right) \approx 79.328^\circ \tag{4}$$