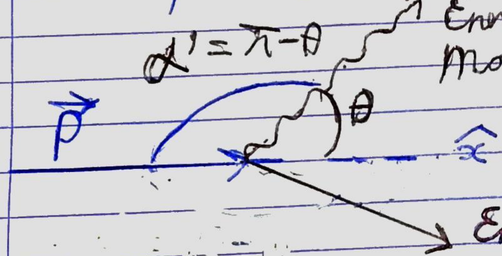
Stage 1: The rocket accelerates to speed βc in the positive x-direction.

From conservation of energy: $E + \gamma \epsilon m c^2 = mc^2$ ①
From conservation of momentum: $E/c = \gamma \epsilon m \beta c$ ②
where ϵm denotes the remaining mass and E the energy of the photons. From ① and ② $\gamma(1+\beta) = 1/\epsilon \Rightarrow$
$(1+\beta)^2/(1-\beta)(1+\beta) = \frac{1}{\epsilon^2} \Rightarrow \beta = (1-\epsilon^2)/(1+\epsilon^2)$
The momentum $P = \gamma \epsilon m \beta c = \frac{1-\epsilon^2}{2} mc$ and the energy $\mathcal{E} = \gamma \epsilon m c^2 = 1+\epsilon^2/2 \; mc^2$, all for the rocket. The emmitted photons aren't relevant.

Stage 2: The rocket dissociates into a piece with mass $1/4\, m$ and speed $4/5\, c$ and a photon (group)



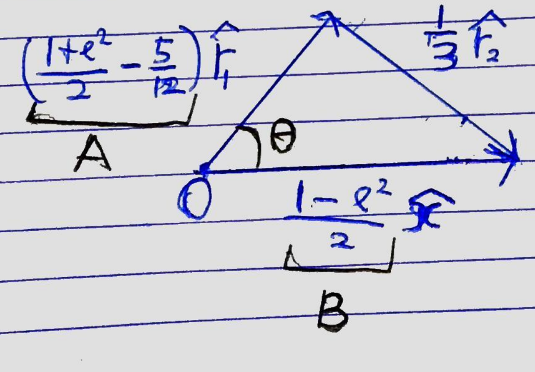
Energy $= E_1$
Momentum $= \vec{P_1} = \frac{E_1}{c} \cdot \hat{r}_1$

Energy $= E_2 = \frac{1}{\sqrt{1-\left(\frac{4}{5}\right)^2}} \cdot \frac{1}{4} mc^2 = \frac{5}{12} mc^2$

Momentum $= \vec{P_2} = \frac{4}{5} \frac{E_2}{c} \hat{r}_2 = \frac{1}{3} mc\, \hat{r}_2$

By conservation of energy $E_1 = \mathcal{E} - E_2 = \left(\frac{1+\epsilon^2}{2} - \frac{5}{12}\right) mc^2$
and hence $\vec{P_1} = \left(\frac{1+\epsilon^2}{2} - \frac{5}{12}\right) mc\, \hat{r}_1$

By conservation of momentum $\vec{P_1} + \vec{P_2} = \vec{P}$, so
$\left(\frac{1+\epsilon^2}{2} - \frac{5}{12}\right) \hat{r}_1 + \frac{1}{3} \hat{r}_2 = \frac{1-\epsilon^2}{2} \hat{x}$



Notice that $A+B = 7/12$, and let $n = \frac{5}{24} - \frac{\ell^2}{2}$, then we have $A = 7/24 - n$ and $B = (7/24 + n)$.
By the cosine rule $A^2 + B^2 - 2AB\cos\theta = (1/3)^2 \Rightarrow$

$$\cos(\theta_2) = \left(2\left(\left(\tfrac{7}{24}\right)^2 + n^2\right) - \tfrac{1}{9}\right) \Big/ \left(2\left(\left(\tfrac{7}{24}\right)^2 - n^2\right)\right) \quad \text{I}$$

Let θ' be the angle ~~θ~~ as seen by the frame which moves with velocity $\beta c \hat{x}$ with respect to the previous frame. Then $\cos(\theta') = \dfrac{\cos(\theta) - \beta}{1 - \beta\cos(\theta)}$ II 2

$$\beta = \frac{1-\ell^2}{1+\ell^2} = \frac{7/24 + n}{17/24 - n} \quad \text{III}$$

From I, II and III: $\cos(\theta') = \dfrac{2304n^2 - 264n - 9}{-2304n^2 - 264n + 119}$ IV

$$\frac{d}{dn}(\cos(\theta')) = -\frac{36n^2 - 15n + 1}{(2304n^2 + 264n - 119)^2} \cdot 33796$$

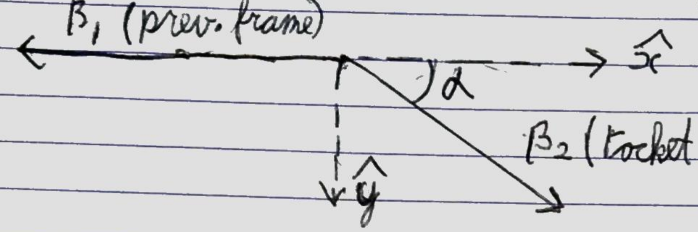
Equating $\frac{d}{dn}$ to 0 gives $n = 1/12$ or $n = 1/3$, which is impossible.
Substituting $n = 1/12$ in IV gives $\cos(\theta') = -5/27$

The minimal value of α is the supplement of this θ'
So the minimal value of α is $\arccos(5/27)$

1 $A+B = 7/12$ means that the locus of O is an ellipse with a major axis of 7/12 and the head and tail of $\vec{P_2}/mc$ as foci.

2 For light $c=1$ so from the Lorentz transforms. $\cos\theta' = \beta'_x = (\beta_x - B)/(1 + \beta_x B) = (\cos(\theta) - B)/(1 + \cos(\theta)B)$, where B denotes the ratio of the velocity of the prime frame and c.

A (mathematically) more pleasing solution is the following:
As proved if $e\,m$ remains then $\beta=(1-e^2)/(1+e^2)$. Lets solve the problem in the frame in which the rocket is at rest when the thrust changes direction. We have the following sceme;



$\beta_1=(1-e^2)/(1+e^2)$ and β_2 (in strict analogy with β_1) is equal to $\left(1-\left(\frac{1/z}{e}\right)^2\right)/\left(1+\left(\frac{1/z}{e}\right)^2\right)=(z^2e^2-1)/(z^2e^2+1)$, where $\frac{m}{z}$ is the mass after stage z has been completed.
At any time 't', the changed-to-euclidean space time vector of the previous frame 'L' is $ct\langle i,-\beta_1,0\rangle$ and that of the rocket 'R' is $ct\langle i,\beta_2\cos\alpha,\beta_2\sin\alpha\rangle$.
In this space we have $\tan(\varphi_r)=\beta_r/i$, wher β_r is the rel. β of L and R and φ_r is the angle between them. From this $\cos(\varphi_r)=(1+\tan^2(\varphi_r))^{-1/2}=(1-\beta_r^2)^{-1/2}=\gamma_r$ and $\sin(\varphi_r)=\cos(\varphi_r)\cdot\tan(\varphi_r)=\gamma_r\beta_r/i$. Since $\gamma_r=\cos(\varphi_r)$, we have
$\gamma_r=\frac{|\langle L,R\rangle|}{|L|\cdot|R|}=|-1-\beta_1\beta_2\cos\alpha|/\sqrt{(-1+\beta_1^2)(-1+\beta_2^2)}=$
$(1+\beta_1\beta_2\cos\alpha)\gamma_1\gamma_2$. From the substitution $\psi=i\varphi$ we get $\cosh(\psi)=\gamma$, $\sinh(\psi)=\beta\gamma$ and $\tanh(\psi)=\beta$, and thus
$\gamma_r=\cosh(\psi_r)=\cosh(\psi_1)\cosh(\psi_2)+\sinh(\psi_1)\sinh(\psi_2)\cos(\alpha)$ I

$\beta_1=\tanh(\psi_1)=(1-e^2)/(1+e^2)\Rightarrow e^2=\frac{1-\tanh(\psi_1)}{1+\tanh(\psi_1)}=\varepsilon^{-2\psi_1}$
$\beta_2=\tanh(\psi_2)=(z^2e^2-1)/(z^2e^2+1)\Rightarrow z^2e^2=\varepsilon^{2\psi_2}$
$z^2=z^2e^2/e^2=\varepsilon^{2(\psi_1+\psi_2)}$ so $\psi_1+\psi_2=\ln(z)$! II

The problem then is: $F(\psi_1,\psi_2)=\cosh(\psi_1)\cosh(\psi_2)+\sinh(\psi_1)\sinh(\psi_2)\cos(\alpha)$, $G(\psi_1,\psi_2)=\psi_1+\psi_2$ minimize 'α' under the constraints $F=\gamma_r$ and $G=\ln(z)$. This is the case when F just touches the line and hence their gradients are parallel i.e.
$$\nabla G \parallel \nabla F \Rightarrow \begin{pmatrix}1\\1\end{pmatrix}\sim\begin{pmatrix}\sinh\psi_1\cosh\psi_2+\cosh\psi_1\sinh\psi_2\cos(\alpha)\\ \cosh\psi_1\sinh\psi_2+\sinh\psi_1\cosh\psi_2\cos(\alpha)\end{pmatrix}\Rightarrow$$

$\sinh(\psi_1-\psi_2)\left(1-\cos(\alpha)\right)=0 \Rightarrow \psi_1=\psi_2$ or $\alpha=0$.

$\alpha=0$ gives $\gamma_r = \cosh(\psi_1+\psi_2) = \cosh(\ln(z)) = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$\psi_1=\psi_2=\frac{1}{2}\ln(z)$ gives $\gamma_r = \cosh^2\psi + \sinh^2\psi\,\cos(\alpha) \Rightarrow$

$1+\sinh^2(\psi)\left(1+\cos(\alpha)\right)=\gamma_r \Rightarrow \cos(\alpha) = \dfrac{\gamma_r-1}{\sinh^2\left(\frac{\ln(z)}{2}\right)}-1$

$$\alpha_{min} = \arccos\left[\frac{(4\gamma_r-2)z-z^2-1}{(z-1)^2}\right]$$

For the specific case where $z=4$ and $\gamma_r=5/3$ we get

$$\boxed{\alpha_{min}=\arccos\left(5/27\right)}$$