The outline of this solution is as follows: I will first prove a general result for rectilinear movement of a photon-propelled ship, after which I will use that result twice - once in the initial rest frame, and once in the rest frame of the spaceship at the moment of turning.

Summary

1 Study of rectilinear rocket motion

Definition

The *rapidity* φ is an adimensional quantity defined through

$$\varphi = \tanh \frac{v}{c}. \tag{1}$$

It can be easily seen that other kinematic and dynamic quantities can also be expressed simply in terms of the rapidity: $\gamma = \cosh \varphi$ and $\gamma v = \sinh \varphi$, so that $E = mc^2 \cosh \varphi$ and $p = mc \sinh \varphi$ (which are the total energy and the absolute value of the momentum of a particle of mass m, respectively).

Now, let us prove a result regarding *rectilinear* rocket motion. Particularly, let us consider a photon-propelled ship with initial mass M_0 that accelerates rectilinearly, starting from rest.

Theorem

At the moment when the ship's mass is M, its rapidity is

$$\varphi = \ln \frac{M_0}{M}. \tag{2}$$

Proof. Let us work in the frame of reference in which the spaceship was at rest initially, and let the total momentum of the ship when its mass is M be P_s.

- Let us consider the total quantity of photons that have been so far emitted by the ship (in what follows, I will name this *the photons*). Since the ship was initially at rest, momentum conservation requires that the total momentum of the photons be also equal to P_s, but in the oposite direction:

$$P_p = P_s \text{ (in absolute value).} \tag{3}$$

- Due to energy conservation,

$$M_0 c^2 = E_s + E_p, \tag{4}$$

  where E_s is the total energy of the ship.

- However, we know that for photons

$$E_p = cP_p, \tag{5}$$

so that introducing this in the above relationship we find that

$$M_0 c^2 = E_s + cP_s. \tag{6}$$

We can write the energy and momentum of the ship as $E_s = Mc^2 \cosh\varphi$ and $P_s = Mc\sinh\varphi$, respectively; therefore, Eq. (6) implies that

$$M_0 c^2 = Mc^2(\cosh\varphi + \sinh\varphi) \Longrightarrow \cosh\varphi + \sinh\varphi = \frac{M_0}{M}. \tag{7}$$

But $\cosh\varphi + \sinh\varphi = e^{\varphi}$, so

$$e^{\varphi} = \frac{M_0}{M} \Longrightarrow \varphi = \ln\frac{M_0}{M}. \tag{8}$$

□

Hence, the speed of the ship will be

$$v = c\tanh\varphi = c\tanh\ln\frac{M_0}{M}. \tag{9}$$

2 Finding an expression of the rocket's final velocity

Let m_1 be the mass of the ship at the end of the first episode of rectilinear motion, and let m_2 be its final mass; we know that

$$m_2 = \frac{1}{4}m. \tag{10}$$

According to Eq. (2), the speed of the ship at the end of the first episode of rectilinear motion will be given by

$$\varphi_1 = \ln\frac{m}{m_1} \Longrightarrow v_1 = c\tanh\varphi_1 = c\tanh\ln\frac{m}{m_1}. \tag{11}$$

Let us now consider the rest frame of the spaceship at the end of the first episode of motion (*i.e.* the frame moving at velocity v_1 along the initial direction of motion of the ship); in what follows, I will name this frame by *the moving frame* and the corresponding quantities will be primed. In this frame, the subsequent movement of the ship is rectilinear, so we can again use Eq. (2) and find that the final speed of this ship in the moving frame is given by

$$\varphi_2' = \ln\frac{m_1}{m_2} \Longrightarrow v_2' = c\tanh\varphi_2' = c\tanh\ln\frac{m_1}{m_2}\text{[1]}. \tag{12}$$

To find the speed of the spaceship in the rest frame, we need to make a transformation from the moving frame. To do this, I will use the Lorentz transformation of the velocity four vector

$$v_\mu = \begin{pmatrix}\gamma & \gamma v_x & \gamma v_y & \gamma v_z\end{pmatrix} \tag{13}$$

and, in particular, I will look at the transformation of γ, assuming that the x-axis of the rest frame is along the initial direction of acceleration of the ship:

$$\gamma = \gamma_{v_1}\left(\gamma' + \frac{v_x' v_1}{c^2}\right)\text{[2]} \tag{14}$$

We know that $\gamma' = \cosh\varphi_2'$; $\gamma_{v_1} = \cosh\varphi_1$; $\gamma_{v_1}v_1 = \sinh\varphi_1$; and $v_x' = v_2'\cos\alpha$ (by the definition of α), or $\gamma' v_x' = c\cos\alpha\sinh\varphi_2'$. Hence,

$$\boxed{\gamma = \cosh\varphi_1\cosh\varphi_2' + \cos\alpha\sinh\varphi_1\sinh\varphi_2'.} \tag{15}$$

[1] To clarify: I considered the motion to start from rest and the initial mass of the ship to be m_1 in this case.

[2] There are a bit too many γ's floating around here, so let me clarify what each of them means. γ is the Lorentz factor of the ship in the rest frame; γ' is the Lorentz factor of the ship in the moving frame; and γ_{v_1} is the Lorentz factor associated with the speed v_1, which appears because v_1 is the speed of the moving frame.

3 Minimization of α

Using the sum and difference formulae for hyperbolic functions:

$$\sinh x \sinh y = \frac{1}{2}(\cosh(x+y) - \cosh(x-y)) \\ \cosh x \cosh y = \frac{1}{2}(\cosh(x+y) + \cosh(x-y)), \tag{16}$$

we can rewrite Eq. (15) as

$$\gamma = \frac{1}{2}(1+\cos\alpha)\cosh(\varphi_1 + \varphi_2') + \frac{1}{2}(1-\cos\alpha)\cosh(\varphi_2' - \varphi_1). \tag{17}$$

Since

$$\varphi_1 + \varphi_2' = \ln\frac{m}{m_1} + \ln\frac{m_1}{m_2} = \ln\frac{m}{m_2}; \varphi_2' - \varphi_1 = \ln\frac{m}{m_1} - \ln\frac{m_1}{m_2} = \ln\frac{mm_2}{m_1^2}, \tag{18}$$

we have

$$2\gamma = (1+\cos\alpha)\cosh\ln\frac{m}{m_2} + (1-\cos\alpha)\cosh\ln\frac{mm_2}{m_1^2}. \tag{19}$$

Now, we know that $\cosh\ln\frac{mm_2}{m_1^2} \geq 1$ (which is a general property of the hyperbolic cosine function), so

$$2\gamma \geq (1+\cos\alpha)\cosh\ln\frac{m}{m_2} + 1 - \cos\alpha \Longrightarrow \cos\alpha \leq \frac{2\gamma - \cosh\ln\frac{m}{m_2} - 1}{\cosh\ln\frac{m}{m_2} - 1}. \tag{20}$$

Since $\cosh\ln\frac{m}{m_2} = \frac{1}{2}(\frac{m}{m_2} + \frac{m_2}{m})$, we have

$$\cos\alpha \leq \frac{4\gamma - \frac{m}{m_2} - \frac{m_2}{m} - 2}{\frac{m}{m_2} + \frac{m_2}{m} - 2} \Longrightarrow \alpha_m = \arccos\frac{4\gamma - \frac{m}{m_2} - \frac{m_2}{m} - 2}{\frac{m}{m_2} + \frac{m_2}{m} - 2}. \tag{21}$$

Numerically, $\gamma = \frac{5}{\sqrt{5^2-4^2}} = \frac{5}{3}$, $\frac{m}{m_2} = 4$, so

$$\boxed{\alpha_m = \arccos\frac{5}{27} \approx 79.328^\circ.} \tag{22}$$

Note: Indeed, this situation does arise, namely for $m_1 = \sqrt{mm_2}$, because in this case $\ln\frac{mm_2}{m_1^2} = 0$ and $\cosh 0 = 1$.