Physics Cup 2022 Problem 3

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Call the rocket Alice.

We solve in the inertial frame R where Alice is at rest when she makes the turn. In this frame, she starts by flying leftward at a speed β_1 and accelerates towards the right. She comes to a rest, changes her direction of thrust by an angle of α, and accelerates towards the top-right, ending with speed β_2 (Figure 1).

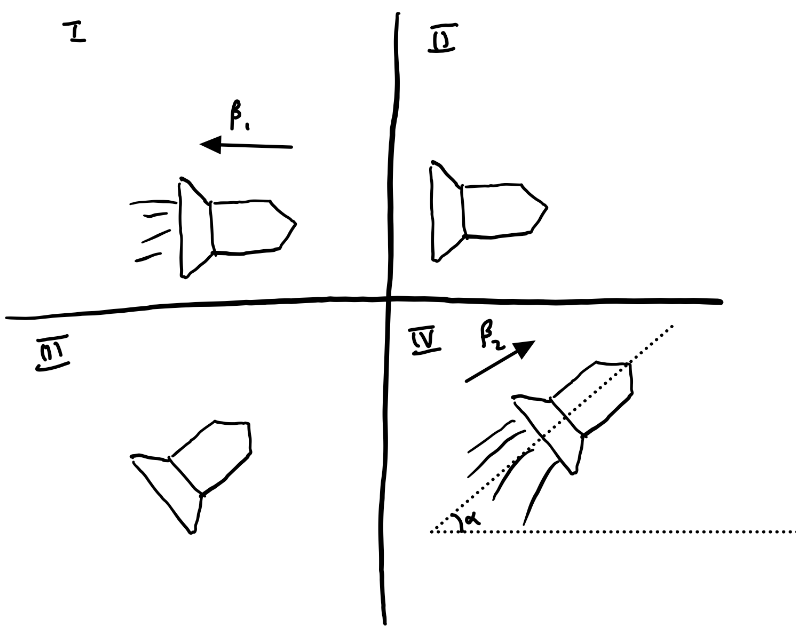

Figure 1: Alice's motion in the frame R

*The LCM of Zed and Eddie (Zed is the name Zhening goes by)

1 Amount of fuel consumption constrains β_1 and β_2 due to momentum and energy conservation

Alice's motion is divided into two phases: before the turn and after the turn.

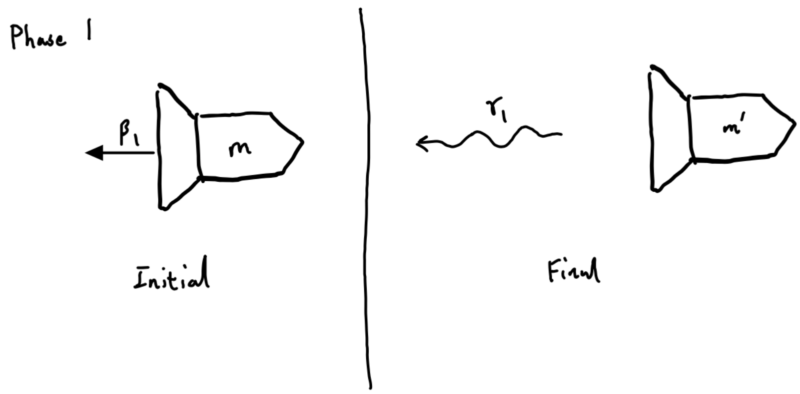

Figure 2: The phase before the turn

During the phase before the turn (Figure 2), energy conservation gives

$$\frac{m}{\sqrt{1-\beta_1^2}} = \frac{E_{\gamma 1}}{c^2} + m',$$

where m' is Alice's mass when she comes to a rest and $E_{\gamma 1}$ is the energy of all photons ejected so far. Momentum conservation gives

$$\frac{m\beta_1}{\sqrt{1-\beta_1^2}} = \frac{p_{\gamma 1}}{c}.$$

Since photons satisfy $E_{\gamma 1}/c^2 = p_{\gamma 1}/c$, we get

$$\begin{aligned}
\frac{m}{\sqrt{1-\beta_1^2}} &= \frac{m\beta_1}{\sqrt{1-\beta_1^2}} + m' \\
\frac{1-\beta_1}{\sqrt{1-\beta_1^2}} m &= m' \\
\sqrt{\frac{1-\beta_1}{1+\beta_1}} &= \frac{m'}{m}. \qquad (1)
\end{aligned}$$

Figure 3: The phase after the turn

Similarly, during the phase after the turn (Figure 3), energy conservation gives

$$m' = \frac{\frac{1}{4}m}{\sqrt{1-\beta_2^2}} + \frac{E_{\gamma 2}}{c^2},$$

where $\frac{1}{4}m$ is Alice's final rest mass and $E_{\gamma 2}$ is the energy of all the photons ejected after the turn. Momentum conservation gives

$$\frac{\frac{1}{4}m\beta_2}{\sqrt{1-\beta_2^2}} = \frac{p_{\gamma 2}}{c}.$$

Using $E_{\gamma 2}/c^2 = p_{\gamma 2}/c$, we get

$$\begin{aligned} m' &= \frac{\frac{1}{4}m}{\sqrt{1-\beta_2^2}} + \frac{\frac{1}{4}m\beta_2}{\sqrt{1-\beta_2^2}} \\ m' &= \frac{1}{4}m\frac{1+\beta_2}{\sqrt{1-\beta_2^2}} \\ \frac{m'}{m} &= \frac{1}{4}\sqrt{\frac{1+\beta_2}{1-\beta_2}}. \end{aligned} \tag{2}$$

Comparing (1) and (2) gives the constraint that β_1 and β_2 have to satisfy:

$$\begin{aligned} \sqrt{\frac{1-\beta_1}{1+\beta_1}} &= \frac{1}{4}\sqrt{\frac{1+\beta_2}{1-\beta_2}} \\ 4 = f(\beta_1, \beta_2) &:= \sqrt{\frac{1+\beta_1}{1-\beta_1}}\sqrt{\frac{1+\beta_2}{1-\beta_2}}. \end{aligned} \tag{3}$$

2 Final speed in ground frame as a function of β_1, β_2, α

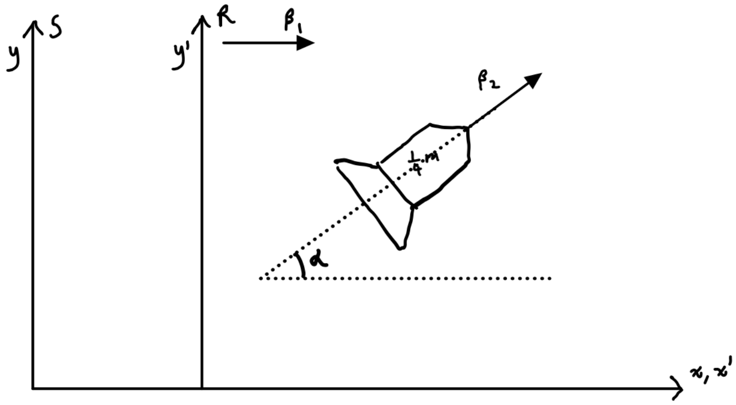

Figure 4: Relationship between frame R and the ground frame

The frame R moves at a speed β_1 towards the right, and Alice's final velocity $\boldsymbol{\beta}_2$ in R makes an angle α with the horizontal (Figure 4). Let's derive the Lorentz transform for velocities, which will give us Alice's final velocity in the ground frame.

With a suitable definition of the origin, Alice's position in R is given by

$$x' = \beta_2 c t' \cos\alpha \qquad\qquad y' = \beta_2 c t' \sin\alpha.$$

We apply the inverse Lorentz transform to get her position and time in the ground frame:

$$\begin{aligned} x &= \frac{x' + \beta_1 c t'}{\sqrt{1-\beta_1^2}} & y &= y' & t &= \frac{t' + \beta_1 x'/c}{\sqrt{1-\beta_1^2}} \\ x &= \frac{\beta_2\cos\alpha + \beta_1}{\sqrt{1-\beta_1^2}} c t' & y &= \beta_2 c t' \sin\alpha & t &= \frac{1 + \beta_1\beta_2\cos\alpha}{\sqrt{1-\beta_1^2}} t'. \end{aligned}$$

So

$$\begin{aligned} v_x &= \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} & v_y &= \frac{dy}{dt} = \frac{dy}{dt'}\frac{dt'}{dt} \\ &= \frac{\beta_2\cos\alpha + \beta_1}{\sqrt{1-\beta_1^2}} c \cdot \frac{\sqrt{1-\beta_1^2}}{1+\beta_1\beta_2\cos\alpha} & &= \beta_2 c \sin\alpha \cdot \frac{\sqrt{1-\beta_1^2}}{1+\beta_1\beta_2\cos\alpha} \\ v_x &= c\frac{\beta_1 + \beta_2\cos\alpha}{1+\beta_1\beta_2\cos\alpha} & v_y &= c\frac{\beta_2\sin\alpha}{1+\beta_1\beta_2\cos\alpha}\sqrt{1-\beta_1^2}. \end{aligned} \tag{4}$$

(4) thus gives Alice's final velocity in the ground frame. Her final speed in the ground frame is given by

$$\begin{aligned} v_f &= \sqrt{v_x^2 + v_y^2} \\ &= c\frac{\sqrt{(\beta_1 + \beta_2\cos\alpha)^2 + \beta_2^2\sin^2\alpha\left(1-\beta_1^2\right)}}{1+\beta_1\beta_2\cos\alpha} \\ &= c\frac{\sqrt{\beta_1^2 + \beta_2^2\cos^2\alpha + 2\beta_1\beta_2\cos\alpha + \beta_2^2\sin^2\alpha - \beta_1^2\beta_2^2\sin^2\alpha}}{1+\beta_1\beta_2\cos\alpha} \\ v_f(\beta_1,\beta_2,\alpha) &= c\frac{\sqrt{\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2\cos\alpha - \beta_1^2\beta_2^2\sin^2\alpha}}{1+\beta_1\beta_2\cos\alpha}. \end{aligned} \tag{5}$$

Note that $v_f(\beta_1,\beta_2,\alpha) = v_f(\beta_2,\beta_1,\alpha)$, which is expected since both represent the relativistic velocity addition of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ that are an angle α apart.

3 The minimal v_f at a given α

Here, we wish to minimize (5) at a given α where β_1, β_2 are constrained by (3).

First, we give a qualitative argument for why v_f cannot be minimized when $\beta_1 = 0$ or $\beta_2 = 0$. When we fix the amount of fuel expended (defined as the decrease in Alice's rest mass), flying in a straight path maximizes her final speed. A straight path is where the turning point is at the start or end of Alice's trajectory, i.e., $\beta_1 = 0$ or $\beta_2 = 0$. Hence, v_f achieves its *maximum* when $\beta_1 = 0$ or $\beta_2 = 0$, meaning that the minimum must be achieved when $\beta_1, \beta_2 > 0$.

Since the minimum is not achieved at the boundary, we can use the first order condition: $\mathrm{d}v_f = 0$ under small deviations $\mathrm{d}\beta_1, \mathrm{d}\beta_2$ constrained by (3). By symmetry of the expressions (5) and (3), we guess that this occurs when $\beta_1 = \beta_2$. We now prove that our guess is correct.

Lemma. *For a differentiable function $g(x,y)$ satisfying $g(a,b) = g(b,a)$ for all a, b,*

$$\partial_x g(c,c) = \partial_y g(c,c).$$

Proof. Differentiate with respect to a the expression $g(a,b) = g(b,a)$ to obtain

$$\partial_x g(a,b) = \partial_y g(b,a).$$

Setting $a = b = c$ yields the desired result. □

We take the total derivative of (3):

$$0 = \frac{\partial f}{\partial\beta_1}\mathrm{d}\beta_1 + \frac{\partial f}{\partial\beta_2}\mathrm{d}\beta_2. \tag{6}$$

Recall that $f(\beta_1,\beta_2)=f(\beta_2,\beta_1)$, so when $\beta_1=\beta_2$, the Lemma gives $\frac{\partial f}{\partial \beta_1}=\frac{\partial f}{\partial \beta_2}$. (6) hence becomes

$$0=\mathrm{d}\beta_1+\mathrm{d}\beta_2. \tag{7}$$

Treating α as a constant, the total derivative of v_f is

$$\mathrm{d}v_f=\frac{\partial v_f}{\partial \beta_1}\mathrm{d}\beta_1+\frac{\partial v_f}{\partial \beta_2}\mathrm{d}\beta_2. \tag{8}$$

Since $v_f(\beta_1,\beta_2,\alpha)=v_f(\beta_2,\beta_1,\alpha)$, we can again apply the Lemma to get $\frac{\partial v_f}{\partial \beta_1}=\frac{\partial v_f}{\partial \beta_2}$. Along with (7) and (8), this yields

$$\mathrm{d}v_f=0,$$

as desired.

Hence, when v_f is minimized, let $\beta_1=\beta_2=\beta$. Then (3) becomes

$$\begin{aligned} 4&=\frac{1+\beta}{1-\beta} \\ 4-4\beta&=1+\beta \\ \beta&=\frac{3}{5}. \end{aligned}$$

The minimal v_f is then

$$v_{fm}(\alpha)=c\beta\frac{\sqrt{2+2\cos\alpha-\beta^2\sin^2\alpha}}{1+\beta^2\cos\alpha}, \tag{9}$$

which decreases with α in the range $0\le\alpha\le\pi$.[1] Note that, for $\frac{4}{5}c$ to be a feasible final speed, v_{fm} must be at most $\frac{4}{5}c$. This requires that α be at least a certain value α_m

[1] Intuitively, for larger α, Alice's trajectory in the ground frame can be more "bent," so v_f can be smaller. Quantitatively, setting $a=\cos\alpha$, we have

$$\frac{2+2\cos\alpha-\beta^2\sin^2\alpha}{(1+\beta^2\cos\alpha)^2}=\frac{2-\beta^2+2a+\beta^2a^2}{(1+\beta^2a)^2}.$$

Setting its derivative w.r.t. a to be negative gives

$$\left(2+2\beta^2a\right)\left(1+\beta^2a\right)-2\left(2-\beta^2+2a+\beta^2a^2\right)\beta^2<0,$$

which simplifies to $1>(2-\beta^2)\beta^2$, which is true as long as $\beta\neq 1$.

satisfying $v_{fm}(\alpha_m) = \frac{4}{5}c$:

$$\begin{aligned}
c\beta \frac{\sqrt{2+2\cos\alpha_m - \beta^2\sin^2\alpha_m}}{1+\beta^2\cos\alpha_m} &= \frac{4}{5}c \\
\frac{3}{5}\frac{\sqrt{2+2A-\frac{9}{25}\left(1-A^2\right)}}{1+\frac{9}{25}A} &= \frac{4}{5} && (A=\cos\alpha_m) \\
\frac{9}{25}\frac{\frac{41}{25}+2A+\frac{9}{25}A^2}{\left(1+\frac{9}{25}A\right)^2} &= \frac{16}{25} \\
9\frac{41+50A+9A^2}{\left(25+9A\right)^2} &= \frac{16}{25} \\
225\left(41+50A+9A^2\right) &= 16\left(25+9A\right)^2 \\
9225+11250A+2025A^2 &= 10000+7200A+1296A^2 \\
729A^2+4050A-775 &= 0 \\
(27A+155)(27A-5) &= 0.
\end{aligned}$$

Since $|A| \leq 1$, we get

$$\cos\alpha_m = A = \frac{5}{27}$$

$$\alpha_m = \arccos\left(\frac{5}{27}\right).$$