Problem No 4 – Solution

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February 24, 2022

"Using straight edge and compass" is abbreviated as USEAC in the following paper.

We first assume that a concave lens was used. The only reason for this assumption is to avoid any possible complications caused by the singularity of the transformation of a convex lens. Note that this assumption is not restrictive in the context of this problem as the same result can be produced by both convex and concave lenses.

We begin our discussion by proving several useful results. First, consider a concave lens with focal length $-f$ situated at the origin, with the optical axis overlapping the x-axis.

Lemma 1. *The image of a straight line in the region $x \geq 0$ is also a straight line in the the same region.*

Proof. Let line L be given by $y = mx + c, x > 0$. Given any point $P = P(x_o, mx_o + c)$ in L, we compute the image $P' = P'(x_i, y_i)$ under the transformation of the lens. Rearranging the focal length formula,

$$\begin{aligned} -\frac{1}{f} &= \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{x_o} - \frac{1}{x_i} \\ \frac{1}{x_o} &= \frac{1}{x_i} - \frac{1}{f} \\ x_o &= \frac{x_i f}{f - x_i} \end{aligned} \tag{1}$$

where we have used the fact that a virtual image has negative image distance. The height of the image can then be obtained via the magnification formula and substituting equation 1.

$$\begin{aligned} y_i &= \frac{x_i}{x_o}(mx_o + c) = mx_i + \frac{cx_i}{x_o} \\ &= mx_i + \frac{c(f - x_i)}{f} \end{aligned} \tag{2}$$

One can see that y_i is a linear function in x_i and is therefore a straight line as claimed. To see that x_i lies in the region $x > 0$ we rearrange equation 1 for x_i to give

$$x_i = \frac{x_o f}{x_o + f}$$

which is clearly positive for $x_o > 0$.

What remains to show is that this still holds true for points lying on the thin lens itself ($x_o = 0$). Extending line L to include the intersection with the y-axis, we see that such a point would have coordinates $P = P(0, c)$. However the transformation of this is trivial, as the image of any point lying on the thin lens would just be the point itself. Substituting $x_i = 0$ into equation 2 we have

$$y_i = m \times 0 + \frac{c(f-0)}{f} = c$$

as required. This concludes the proof.

Corollary 2. *Given that the image is a straight line in the region $x \geq 0$, the object must also be a straight line in the same region.*

Proof. If the mapping from a line L to its image L' is a bijection, then the above statement goes without question. Analysing equation 2, if L has slope m and y-intercept c, the image L' will have slope $m_i = m - c/f$ and y-intercept $c_i = c$. Thus varying m or c will always vary m_i or c_i, hence the mapping is injective. One can also see that any combination of m_i and c_i is achievable by solving for m and c, thus the mapping is surjective. Therefore the mapping is bijective and the proof is concluded.

We now proceed with the solution. Firstly, lemma 1 suggests that points $A'B'C'D'$ must be co-linear, or else the problem would be unsolvable. Second, corollary 2 suggests that as long as $A'B'C'D'$ all lie on the same side of the lens (or just touches the lens in the case of $x = 0$), the angle of the lens does not affect the co-linearity of the original four points. That is, the only problem is to choose the angle of the lens such that the original points are evenly-spaced out. We observe that this also means that the perpendicular projections of the original points onto the lens have to be equally-spaced.

Hence, We can reduce the problem to the following statement: Given a focal point F not co-linear with $A'B'C'$, we can trace rays $R_A R_B R_C$ from F that pass through each of these points. Noting our observation from earlier, we have to choose a plane for the lens such that the intersection of the lens with $R_A R_B R_C$ are evenly-spaced. We therefore propose the following procedure.

First, the focal point F is chosen to better suit our needs. In particular, we choose F such that $\angle A'FC' = \pi/2$. This is easily achieved USEAC by taking the midpoint M of $A'C'$, then drawing a circle C centered at M with radius $A'M$. Then it is a well known circle theorem which guarantees that any point F chosen along the circumference of C will have the property specified above.

Second, since the exact position of the lens is insignificant (as long as $A'B'C'D'$ are all on the same side of the lens), we choose that the plane of the lens G passes through A'. The validity of this choice is proved in lemma 1 via the edge case of $x = 0$. The precise angle of G is chosen such that the intersection B'' of G and R_B forms an isosceles triangle $A'B''F$ with $A'F$ as its base. This again can be done easily USEAC by first finding the perpendicular bisector h of $A'F$, then marking the intersection of h and R_B which is B''.

Marking the intersection of G and R_C as C'', we now claim that $|A'B''| = |B''C''|$ which validates our choice for G. We prove this claim geometrically,

$$
\begin{aligned}
&h // R_C && \text{(supplementary angles)}\\
&\angle MB''B' = \angle B'FC' && \text{(alternating angles)}\\
&\angle MB''B' = \angle MB''A' && \text{(isosceles triangle)}\\
&\angle MB''A' = \angle B''C''F && \text{(corresponding angles)}\\
&\Rightarrow \angle B'FC' = \angle MB''B' = \angle MB''A' = \angle B''C''F && \\
&\Rightarrow \Delta B''C''F && \text{is isosceles.}\\
&\Rightarrow |B''F| = |B''C''| && \text{(isosceles triangle)}\\
&\Rightarrow |A'B''| = |B''F| = |B''C''| && \text{(isosceles triangle)}
\end{aligned}
$$

as required, thus the choice of G is appropriate.

Using G and the existing perpendicular projections onto the plane of the lens, we can determine the perpendicular projection of D onto G. This can be done USEAC by drawing a circle centered at C'' with radius $B''C''$, and marking the other intersection D'' that the circle makes with G.

Finally, we draw a line from D'' to F and mark the intersection with the line that passes through $A'B'C'$. This is the image D' of D. Reading off Geogebra, this point is calculated to be $D' = D'(7.115, 2.898)$.