Physics Cup 4^{th} Problem

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First of all we see that a thin lens maps lines in lines: to prove this for simplicity let's consider a system in which the lens is in the origin and the y-axis is parallel to the optical axis. So in this configuration the formula of thin lens is:

$$\frac{1}{f} = \frac{1}{x} - \frac{1}{x'} \Rightarrow x' = \frac{xf}{f-x} \tag{1}$$

Moreover the formula for the transformation of y-coordinate is:

$$y' = \frac{y}{x}x' = \frac{yf}{f-x} \tag{2}$$

where f is the focal length of the lens, x, y the coordinates of the object and x', y' the coordinates of his image.
Now let's consider a generic linear object of equation $y = mx + q$, we can find the equation of his image by substituting the inverse of (eq.1,eq.2) and after some calculations it can be found that:

$$y' = x'(m + \frac{q}{f}) + q \tag{3}$$

which is indeed the equation of a straight line.
Now two important observations:

1. the y-intercept of the line doesn't change: y(0)=y'(0)=q, this means that there is a point (in this case it is (0;q)) in which the object, the image and the line passing through the optical axis intersect (from now on I will call this point O);

2. the slope of the image is $m' = m + \frac{q}{f}$, this means that the only role of the focal length is changing the slope of the image in the range $(-\infty, \infty)$.

Since what I proved is invariant under coordinate transformations those properties will be true even in our problem in which the lens has a generic position and orientation.
Finally the problem can be solved in this way: we pick the point O on the line through $A'B'C'$, then we draw a random line s passing through O which will be the object (from observation 2 we know that all the slopes are physically possible and of course A,B,C,D will lay on that line) and we pick a random point D_1 on the plane which will be the center of the thin lens (and hence the line through O and D_1 is the optical axis).
A, B, C can be found by intersecting s with lines through $A'D_1$, $B'D_1$, $C'D_1$ (because the optical ray passing through the center of the lens is not deviated).

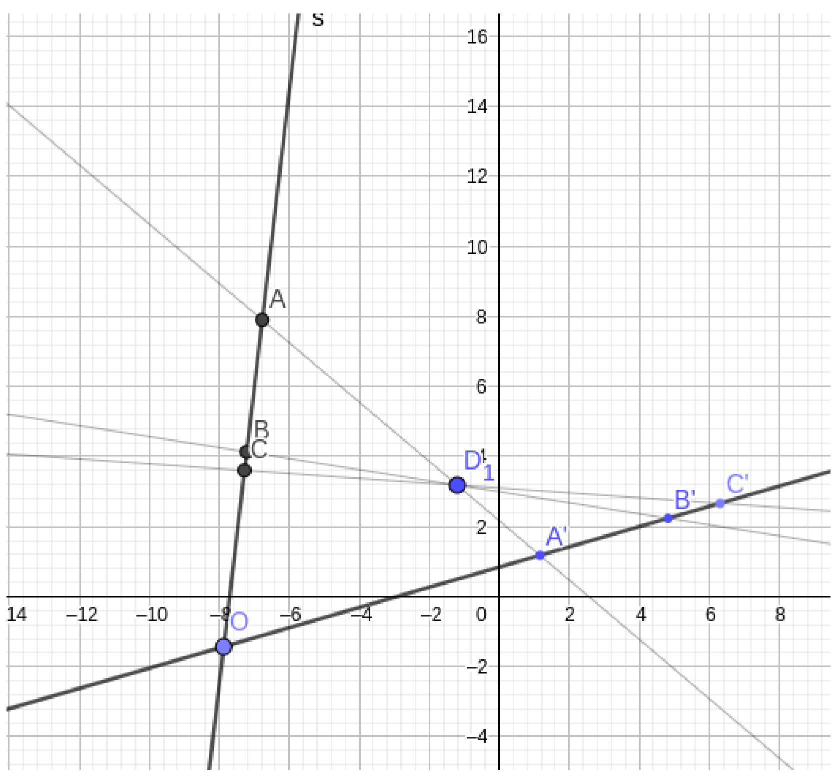

Figure 1: An example of construction of A, B, C

So the problem can be reformulated in this way: we want to find a configuration of O,s and D_1 such that $|AB| = |BC|$, then we can find the point D on the line s such that $|CB| = |CD|$ and consequently D' by intersecting the image line with the line through DD_1.
An easy way to find such a configuration is picking $O \equiv A' \Rightarrow A' \equiv A$,then we draw s randomly the plane (but neither parallel nor orthogonal to the line through $A'B'C'$) and we define the line h orthogonal to s and passing through B'. So we take B as the intersection between h and s, C on s such that B is the midpoint of AC.
Finally we construct D_1 as the intersection between h and the line through CC'. ($AA'D_1$ also lay on the same line since $A \equiv A'$)

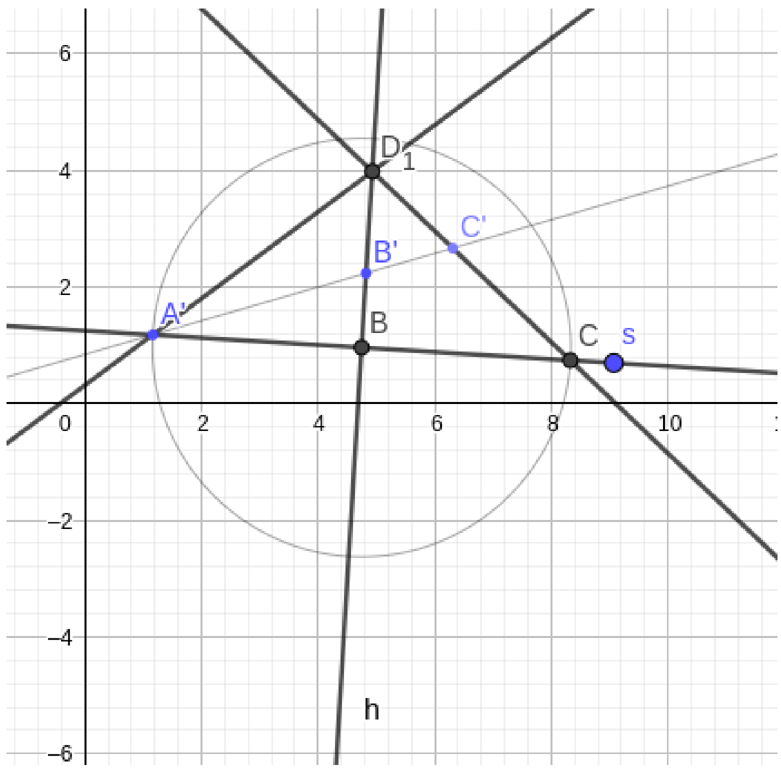

Figure 2: Example of a valid construction of A, B, C

That's a valid configuration since $|AB| = |BC|$, to conclude we take D on s such that C is the midpoint of BD and D' is the intersection between DD_1 and the line through $A'B'C'$.
With some algebra (calculations are very easy if we take s parallel to the x-axis) or with geogebra it can be easily found that D' has coordinates (up to 3 decimal digits):

$$(D'_x; D'_y) = (7.115; 2.898) \tag{4}$$

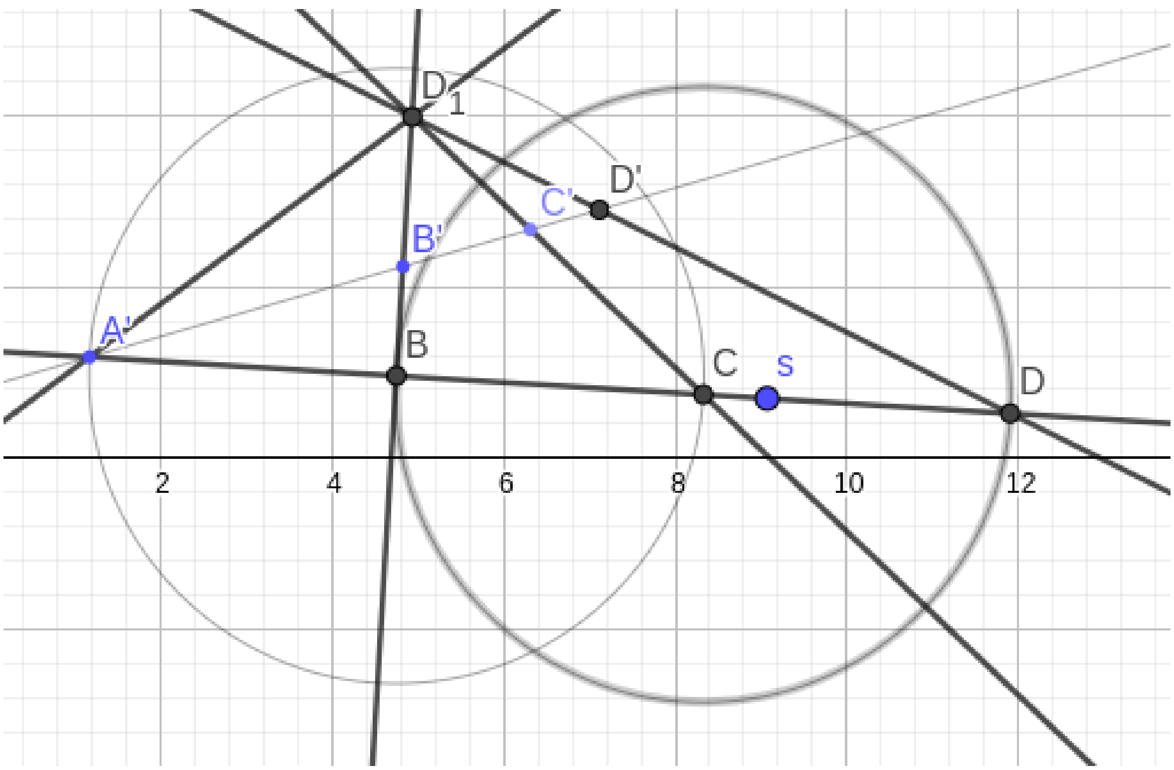

Figure 3: Example of a full construction