Physics Cup - Problem 4

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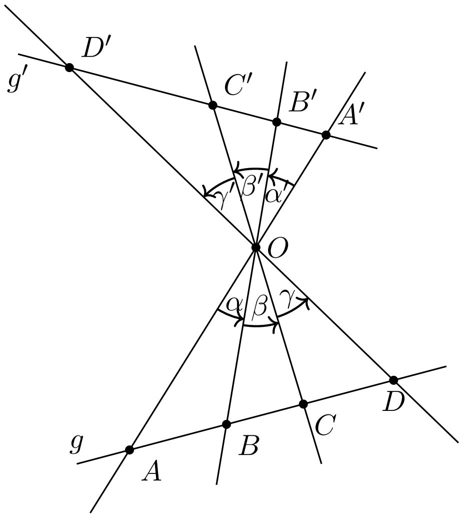
Assume that the lens has focal length f, is positioned on the y-axis, and has its center at the origin. From the thin lens equation follows that the image of a point (x, y) in the right half plane is given by $\left(\frac{-fx}{x-f}, \frac{-fy}{x-f}\right)$. This is a linear fractional transformation, which implies that the image of an object is given by a projective transformation. In particular, this shows that the image of a line is again a line. Hence we can conclude that D' has to lie on the line connecting the points A', B', and C'.

Lemma 1.1. *We have $(A, C; D, B) = (A', C'; D', B')$, where $(A, C; D, B) := \frac{\overline{AD} \cdot \overline{CB}}{\overline{AB} \cdot \overline{CD}}$ is the cross ratio of the four given points, if both tuples are related via a central projection.*

Proof



Let O be the center of the central projection. Furthermore, let g be the line through A, B, C, D. By definition, we have:

$$(A, C; D, B) = \frac{\overline{AD} \cdot \overline{CB}}{\overline{AB} \cdot \overline{CD}} = \frac{\frac{h}{2}\overline{AD} \cdot \frac{h}{2}\overline{CB}}{\frac{h}{2}\overline{AB} \cdot \frac{h}{2}\overline{CD}} = \frac{\Delta ADO \cdot \Delta CBO}{\Delta ABO \cdot \Delta CDO} \tag{1}$$

where h is the distance from O to g. Let α, β, γ be the oriented angles as shown in the figure. Then we can rewrite the cross ratio as follows:

$$\begin{aligned}(A, C; D, B) &= \frac{|\overline{OA}||\overline{OD}| \sin(\alpha + \beta + \gamma) \cdot (-|\overline{OB}||\overline{OC}| \sin \beta)}{|\overline{OA}||\overline{OB}| \sin \alpha \cdot |\overline{OC}||\overline{OD}| \sin \gamma} \\ &= -\frac{\sin(\alpha + \beta + \gamma) \cdot \sin \beta}{\sin \alpha \cdot \sin \gamma}\end{aligned} \tag{2}$$

Using $\alpha = \alpha', \beta = \beta', \gamma = \gamma'$ and the symmetry of the configuration proves the claim.
□

Since the four points A, B, C, D satisfy $|\overline{AB}| = |\overline{BC}| = |\overline{CD}| =: a$, we can conclude that their cross ratio is given by $(A, C; D, B) = \frac{3a \cdot (-a)}{a \cdot a} = -3$.[1] From Lemma 1.1 follows that the point D' has to satisfy $(A', C'; D', B') = -3$. Therefore the problem reduces to constructing a point on a given line such that the points have a given integral cross ratio.
Before proceeding with the construction, we prove a useful claim about cross ratios.

Lemma 1.2. *Let U, V, W, X_n be pairwise distinct collinear points such that $(U, W; X_n, V) = -n$ for $n > 0$. Then the following holds*

$$(X_n, W; X_{n+1}, X_{n-1}) = -1$$

with the convention $X_0 = U$ and $X_{-1} = V$.

Proof

To prove this claim, we embed the line in the $\mathbb{R}^2$ and introduce a coordinate system such that U has the coordinates $\begin{pmatrix} u \\ 1 \end{pmatrix}$, V has the coordinates $\begin{pmatrix} v \\ 1 \end{pmatrix}$, ... Notice that the oriented distance between two points can be written as

$$\overline{PQ} = q - p = \det \begin{pmatrix} q & p \\ 1 & 1 \end{pmatrix} =: [Q, P] \tag{3}$$

This shows that we can express all cross ratios in terms of determinants of some 2×2 matrices.
Define $M = \begin{pmatrix} w(v-u) & u(w-v) \\ v-u & w-v \end{pmatrix}$. We have $\det(M) \neq 0$ because all points are pairwise distinct and

$$M \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u \\ 1 \end{pmatrix} \cdot (w - v) \tag{4}$$

$$M \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} w \\ 1 \end{pmatrix} \cdot (v - u) \tag{5}$$

$$M \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} v \\ 1 \end{pmatrix} \cdot (w - u) \tag{6}$$

Since the numerator and denominator of the cross ratio are both homogeneous of degree 2, we can conclude that scaling a vector by some arbitrary number $\lambda \neq 0$ doesn't change the cross ratio. If all points are multiplied by the same invertible matrix, the cross ratio is also invariant because the determinant is multiplicative. Hence after multiplying all points with M^{-1}, equations (4)-(6) and the scaling argument show that we can w.l.o.g. assume that $U = \binom{0}{1}, V = \binom{1}{1}, W = \binom{1}{0}, X_n = \binom{x_n}{1}$ holds. The coordinates of the points X_n are given as:

$$-n = (U, W; X_n, V) = \frac{[X_n, U][V, W]}{[V, U][X_n, W]} = \frac{x_n \cdot (-1)}{1 \cdot (-1)} = x_n \tag{7}$$

[1]Note that the definition uses oriented segments. A negative value is therefore perfectly fine.

Therefore the claim can be rewritten as follows:

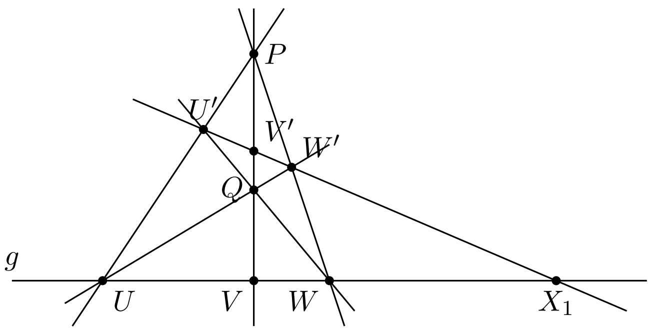
$$(X_n, W; X_{n+1}, X_{n-1}) = \frac{[X_{n+1}, X_n][X_{n-1}, W]}{[X_{n-1}, X_n][X_{n+1}, W]} = \frac{1 \cdot (-1)}{(-1) \cdot (-1)} = -1 \tag{8}$$

□

The above proof shows that the 3 collinear points U, V, and W define a projective scale, where we can identify U with the point 0, V with the point 1, and W with the point at infinity. Furthermore, the proof also shows that if we are able to construct the point X_1, then we can repeat the same construction to obtain any point X_n.
To find the point X_1 we use the following construction:

1. Draw a line g through U, V and W.
2. Let W' be an arbitrary point not on g and draw a line connecting W' and W.
3. Choose an arbitrary point Q on $\overline{UW'}$ ($Q \neq U, W'$) and let P be the intersection point of $\overline{WW'}$ and $\overline{VQ}$.
4. Let U' be the intersection point of $\overline{UP}$ and $\overline{WQ}$.
5. Then X_1 is the intersection point of g and $\overline{U'W'}$.

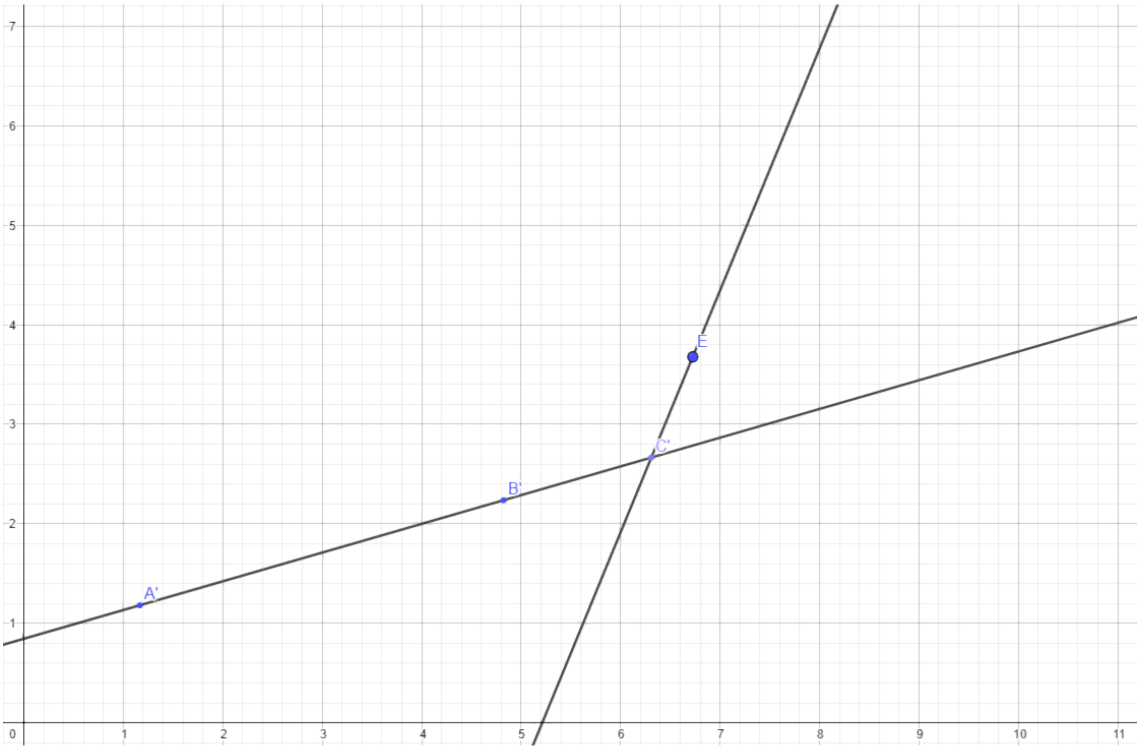
Furthermore, define V' to be the intersection of $\overline{QV}$ and $\overline{U'W'}$.



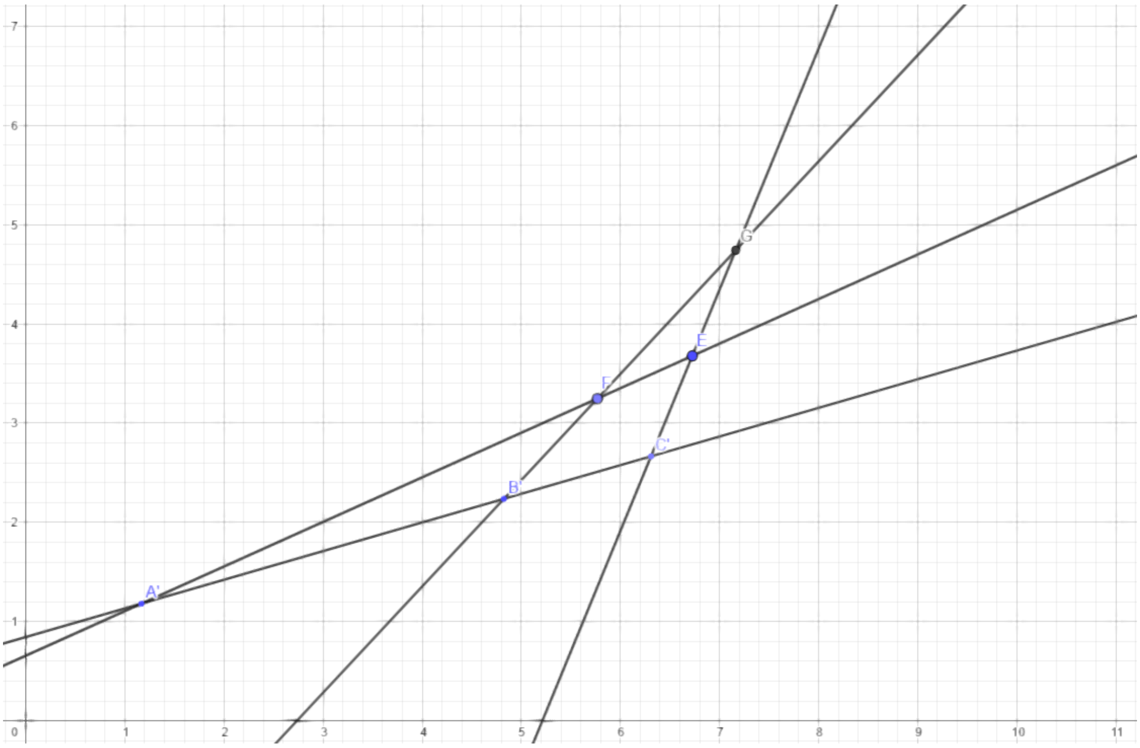
To show that indeed $(U, W; X_1, V) = -1$ holds we repeatedly use Lemma 1.1.

$$\begin{aligned}
(U, W; X_1, V) &= (U', W'; X_1, V') && \text{use } P \text{ as center} && (9)\\
&= (P, Q; V, V') && \text{use } U \text{ as center} && (10)\\
&= (U, W; V, X_1) && \text{use } U' \text{ as center} && (11)\\
&= \frac{1}{(U, W; X_1, V)} && \text{by definition} && (12)
\end{aligned}$$

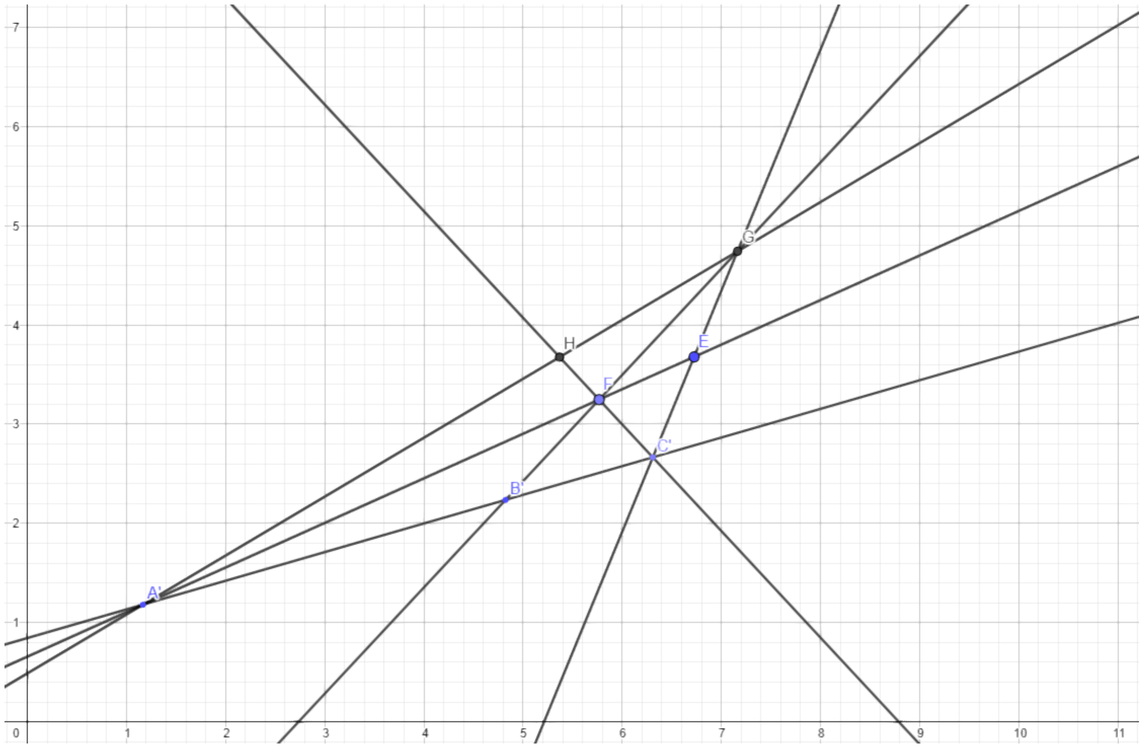
Hence we must have $(U, W; X_1, V) = \pm 1$. Because W always lies between V and X_1 we can discard the positive solution. Repeating this construction allows us to find the sought point $\boxed{D' = X_3 = (7.11544, 2.89808)}$.



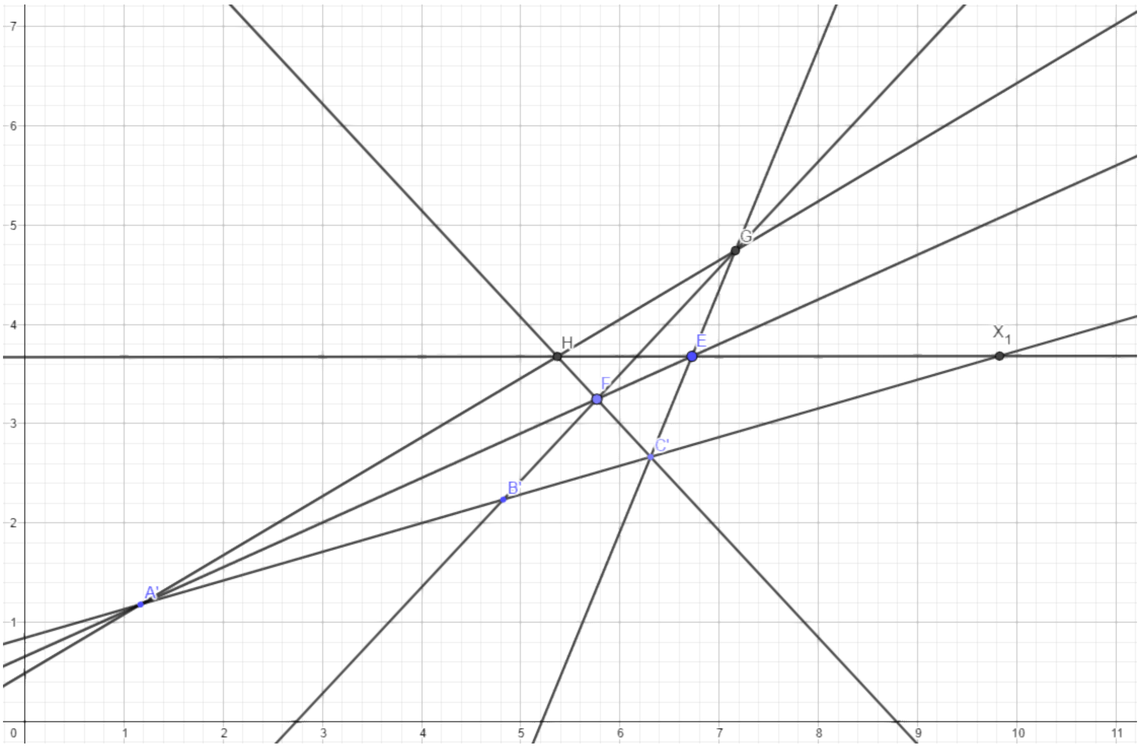
Steps 1 and 2 of the construction.



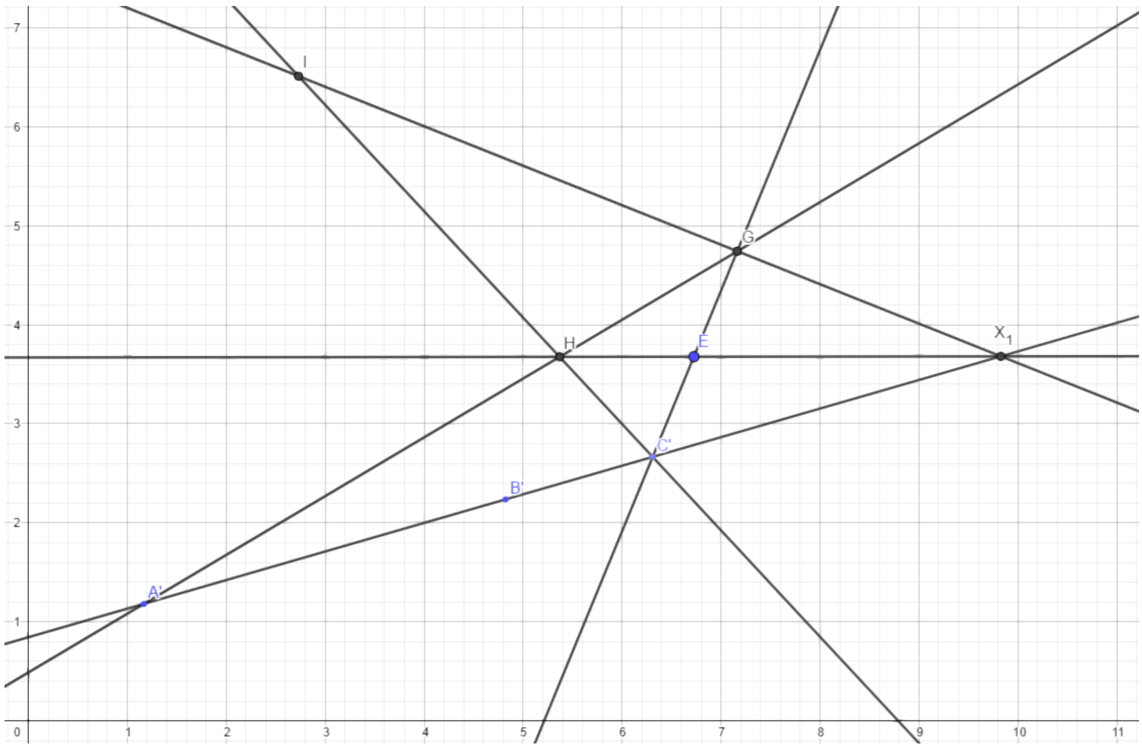
Step 3 of the construction.



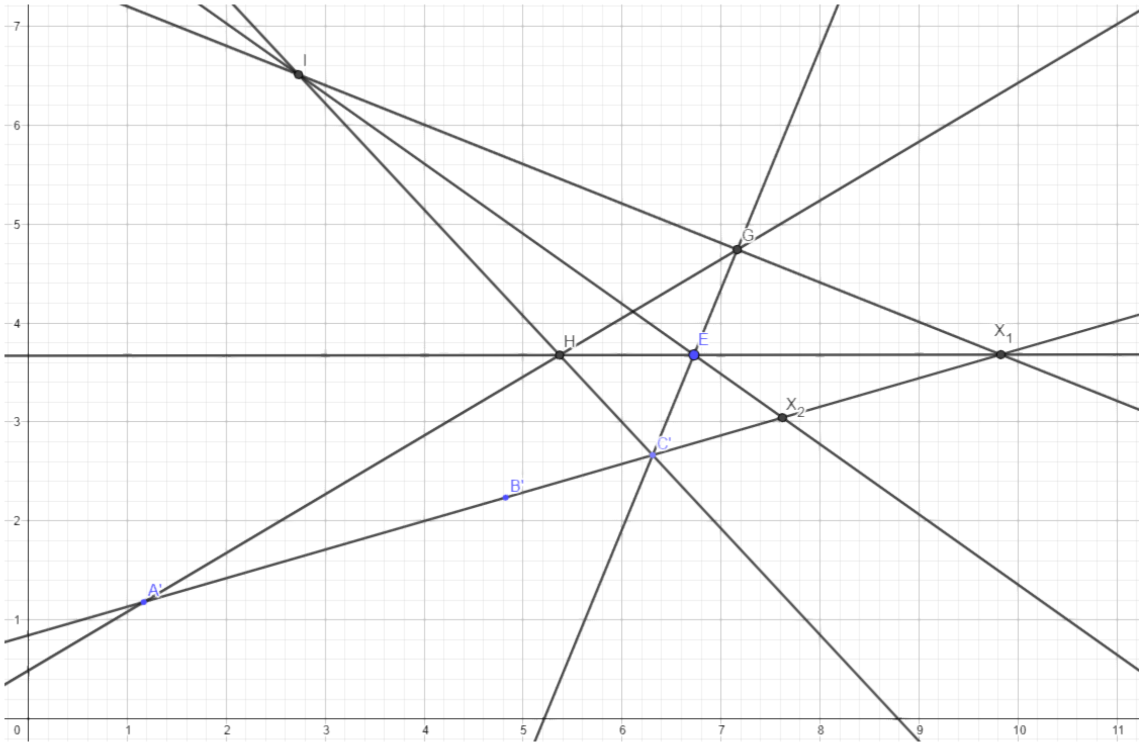
Step 4 of the construction.



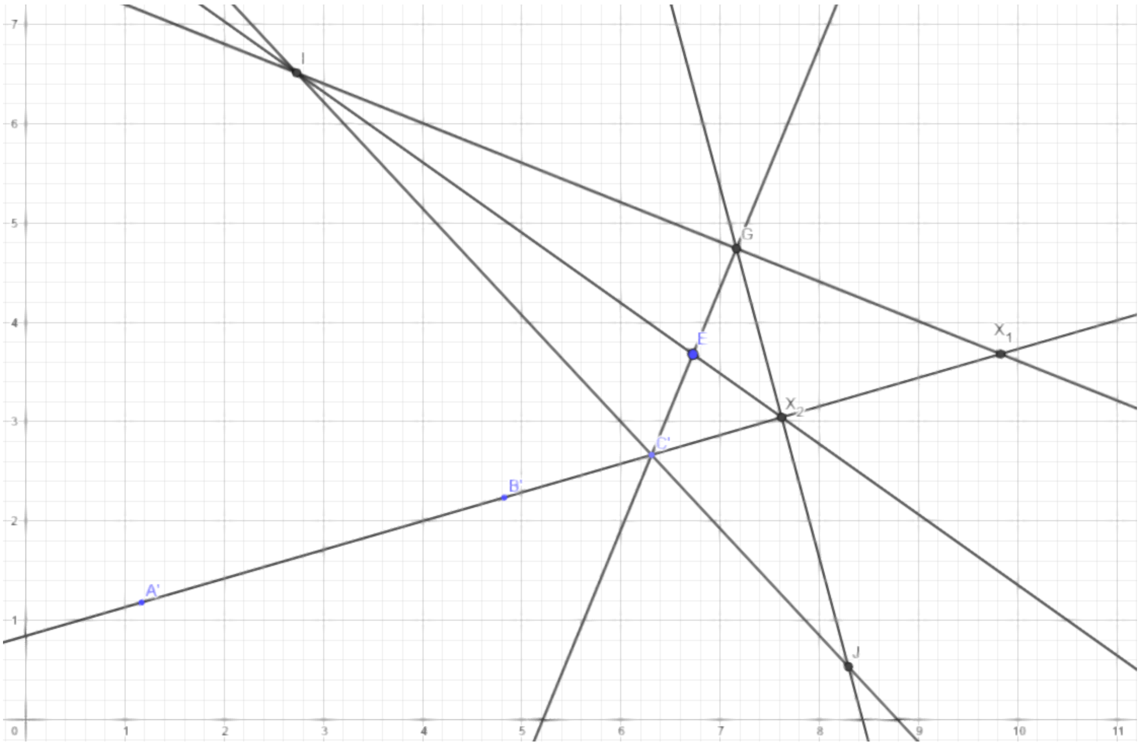
Step 5 of the construction. The first point X_1 is found.



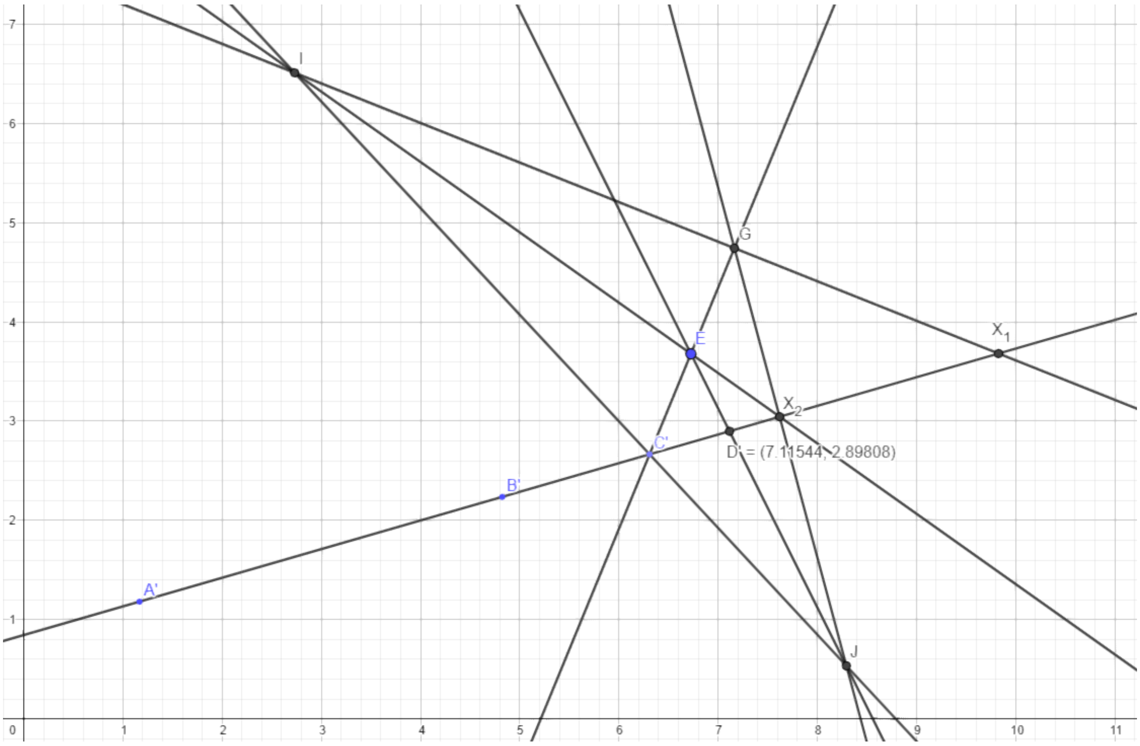
Hide unnecessary lines and reuse already constructed points. ⇒ We can jump to step 4 of the construction.



Step 5 of the construction. The second point X_2 is found.



Hide unnecessary lines and reuse already constructed points. ⇒ We can jump to step 4 of the construction.



Step 5 of the construction. The last point $D' = X_3$ is found.