Physics Cup 2022 Problem 4

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1 Solution

1.1 Key Lemmas

Lemma 1.1. *The image of a straight line L is a straight line L'.*

Proof. Consider a light ray along L. Then the image of L must coincide with the refracted ray, which is a straight line. □

Lemma 1.2. *For 4 collinear points A, B, C, D, the transformation that takes them to their images A', B', C', D' is a perspectivity.*

Proof. A', B', C', D' are also collinear by Lemma 1.1, and the lines $AA', \ldots, DD'$ all pass through the optical center O. Thus, the transformation that takes $A \to A', \ldots, D \to D'$ is a perspectivity through O. □

Lemma 1.3. *Let A, B, C, D be 4 equally spaced points on a line (in that order). Then their images A', B', C', D' lie on a line in that order, and*

$$\frac{A'D'}{C'D'} = 3\frac{A'B'}{B'C'}.$$

Proof. Since a perspectivity is a projective transformation, by Lemma 1.2, the cross-ratio $(A', C'; D', B')$ equals the cross-ratio $(A, C; D, B)$:

$$\frac{A'D'}{C'D'} \Big/ \frac{A'B'}{C'B'} = \frac{AD}{CD} \Big/ \frac{AB}{CB} = 3/(-1) = -3$$

$$\frac{A'D'}{C'D'} = -3\frac{A'B'}{C'B'} = 3\frac{A'B'}{B'C'},$$

where we used signed ratios. □

1.2 Construction of D'

1. Draw line L' through A', B', C'.
2. Draw circle O_1 centered at C' with radius $C'A'$.

3. Pick an arbitrary point P_1 on O_1.
4. Draw circle O_2 centered at C' with radius $B'A'$. Let point P_2 be the intersection of O_2 with L' (closer to B').
5. Draw circle O_3 centered at P_2 with radius $B'A'$. Let point P_3 be the intersection of O_3 with L' (further from C').
6. Draw circle O_4 centered at P_3 with radius $B'A'$. Let point P_4 be the intersection of O_4 with L' (further from C').
7. Draw circle O_5 centered at P_4 with radius $B'C'$. Let point P_5 be the intersection of O_5 with L' (closer to C').
8. Draw line L_1 that passes through P_5 and P_1.
9. Draw line L_2 parallel to P_5P_1 that passes through B'. Let P_6 be the intersection of L_2 and $C'P_1$.
10. Draw circle O_6 centered at C' with radius $C'P_6$. Let D' be the intersection of O_6 with L' (further from B').

1.3 Proof of Construction

According to Lemma 1.1, D' must be on the line L'. According to Lemma 1.3, we must have that D' is on the side of C' opposite to A' and B' with

$$C'D' = \frac{B'C' * A'C'}{3 * A'B' - B'C'}$$

Steps 1-3 of the construction produce a point P_1 s.t. $P_1C' = A'C'$. Steps 4-7 produce the point P_5 that is on L' with $P_5C' = 3 * A'B' - B'C'$. Steps 8-9 produce the point P_6 s.t. $B'C'P_6$ and $P_5C'P_1$ are similar triangles. This means that

$$C'P_6 = \frac{C'P_1 * C'B'}{C'P_5} = \frac{B'C' * A'C'}{3 * A'B' - B'C'}$$

Finally, step 10 produces the point D' with D' on L' and

$$C'D' = \frac{B'C' * A'C'}{3 * A'B' - B'C'}$$

as desired.