The outline of this solution is as follows: first, I will prove and define some general results and concepts regarding geometrical optics and geometry; afterwards, I will use these results in the particular case of this problem.

Contents

1 General considerations

First, a theorem regarding images formed through lenses.

Theorem

Let x_1 be the distance between the object and the object plane of a lens, and let x_2 be the distance between the image and the image focal plane. If f is the focal length of the lens, then

$$x_1 x_2 = f^2. \tag{1}$$

Proof. According to the thin lens equation,

$$\frac{1}{f+x_1} + \frac{1}{f+x_2} = \frac{1}{f} \Longrightarrow 2f^2 + f(x_1+x_2) = f^2 + f(x_1+x_2) + x_1x_2 \Longrightarrow x_1x_2 = f^2. \tag{2}$$

□

Now, let us define a certain geometrical concept and describe a way of constructing it.

Definition

Let A, B, C, and D be four collinear points, in this order. The points are said to form a *harmonic range* if and only if

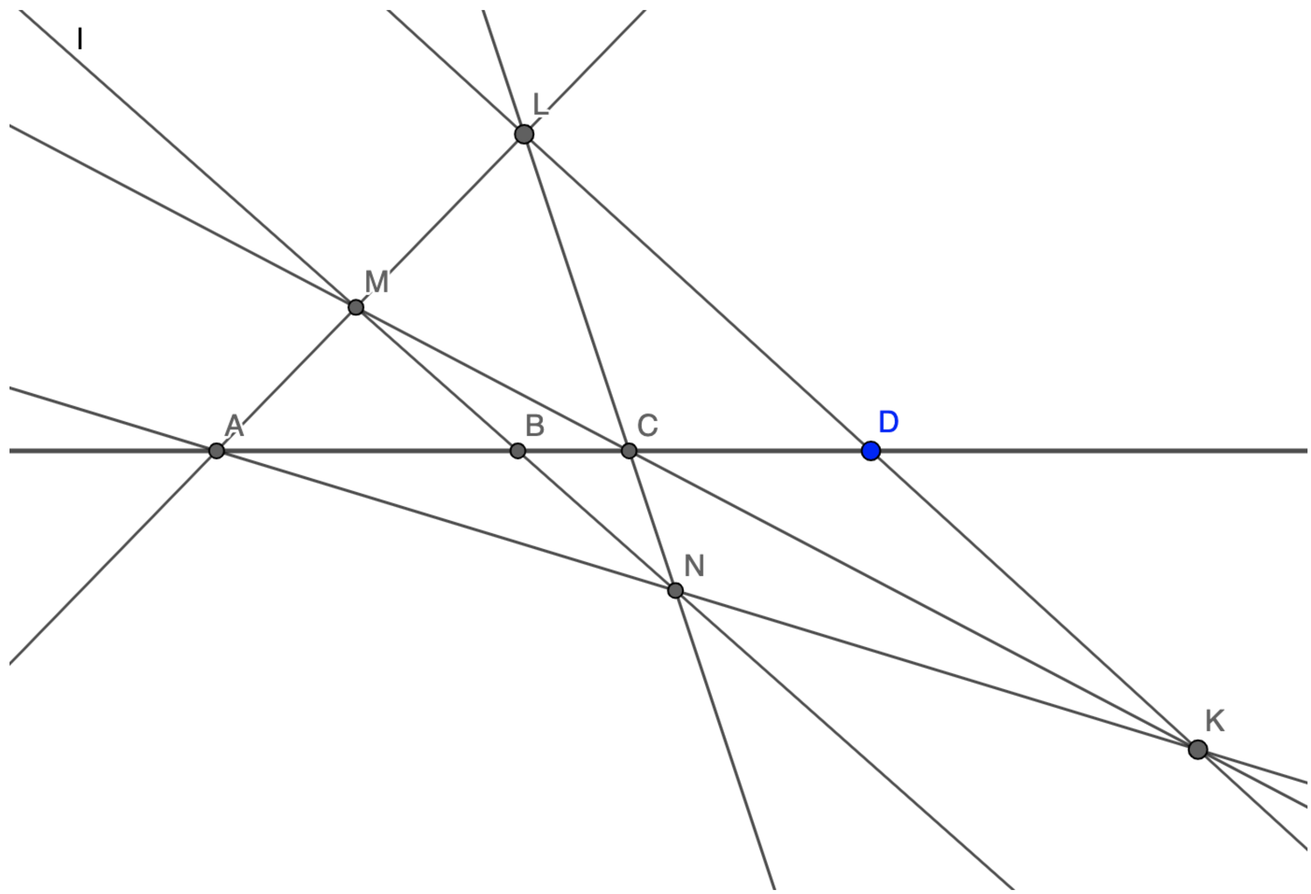
$$AD \cdot BC = AB \cdot CD. \tag{3}$$

In this case, then D is said to be the *harmonic conjugate* of B with respect to A and C; C is said to be the harmonic conjugate of A with respect to B and D; and so on.

Theorem

Let there be three collinear points A, B and C. To construct the harmonic conjugate D of B with respect to A and C, the following construction can be used:

- Pick any arbitrary point L that is non-collinear with A, B and C;
- Pick a line l that passes through D and that intersects LA in M and LC in N;
- Let $K \in AN \cap CM$;
- Construct D as the intersection of the line AB with LK.



2 The actual problem

- The image of any line is a line, and the original line and its image meet at the plane of the lens. As expected, A', B' and C' are collinear - and D' will lie on this same line.
- Let the intersection of the image focal plane of the lens and this line be E. (We do not yet know where E is.) Let $x_{A'}$ be the distance between the image focal plane and the point A', and the same for the other image points too. Then, these distances will be proportional to the distances between the respective points and E (the only difference is a factor that relates to the orientation of the focal plane with respect to the line):

$$\frac{x_{A'}}{EA'} = \frac{x_{B'}}{EB'} = \frac{x_{C'}}{EC'} = \frac{x_{D'}}{ED'}. \tag{4}$$

- Let us denote the distances between the *object* focal plane and the points A, B, C and D by x_A, x_B, x_C and x_D, respectively. According to the theorem proved in Section 1, we know that

$$x_A x_{A'} = f^2 \Longrightarrow x_{A'} = \frac{f^2}{x_A}, \tag{5}$$

and the same for the other three points.

- B is the midpoint of AC - hence,

$$x_B = \frac{x_A + x_C}{2}. \tag{6}$$

Using Eq. (5) and its equivalents for the other points and simplifying, this means that

$$\frac{2}{x_{B'}} = \frac{1}{x_{A'}} + \frac{1}{x_{C'}} \Longrightarrow \frac{x_{A'} - x_{B'}}{x_{B'} - x_{C'}} = \frac{x_{A'}}{x_{C'}}; \tag{7}$$

using the above-shown proportionality, we can write this as

$$\frac{EA' - EB'}{EB' - EC'} = \frac{EA'}{EC'} \Longrightarrow \frac{A'B'}{B'C'} = \frac{EA'}{EC'} \Longrightarrow A'B' \cdot EC' = B'C' \cdot EA'. \tag{8}$$

This proves that A', B', C' and E, in this order, form a harmonic range - and, hence, **we can construct** E **as the harmonic conjugate of** B' **with respect to** A' **and** C'.

- Similarly, C is the midpoint of BD; hence, according the an analogous reasoning, we find that

$$B'C' \cdot ED' = C'D' \cdot EB', \tag{9}$$

so that B', C', D' and E, in this order, form a harmonic range, and **we can construct** D' **as the harmonic conjugate of** B' **with respect to** C' **and** E.

Hence, to construct D', we must first construct E. However, as a simplification, both constructions can be done using almost the same set of objects.

To do this, let L be an arbitrary point and l a line through B' - in the GeoGebra file, this line is colored blue. Let the intersection points of LA' and LC' with l be M and N, respectively, and let the intersection of $A'N$ and $C'M$ be K. Then, E will lie at the intersection between $A'B'$ and LK.

Further, let the intersection of LE with l be P, and let the intersection of EN and $C'P$ be Q. Then, D' will lie at the intersection between $A'B'$ and LQ. The coordinates of D are **(7.115, 2.898)**, and the GeoGebra file with the whole construction is attached separately.