Physics Cup 2022 Problem 5

Eddie Chen and Zhening Li

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Let side H be the side of the satellite facing the sun, and side L be the side facing away from the sun. We claim that the most general such satellite is one painted uniformly, with emissivity ϵ_H for side H and ϵ_L for side L. Note that $0 < \epsilon_H, \epsilon_L \leq 1$. Let the temperature of side H be T_H and that of side L be T_L. Let σ be the Stefan-Boltzmann constant.

In order for these temperatures to be maintained, the heat flux into side H must equal the outgoing heat flux; similarly, this is true for L. Let Q_H be the heat flux from side H into the inside of the satellite, and let Q_L be the heat flux from the inside of the satellite into side L. Thus, using the Stefan-Boltzmann law for radiation, we have

$$Q_H = \epsilon_H (AS - \sigma A T_H^4)$$
$$Q_L = \epsilon_L \sigma A T_L^4,$$

where the power that gets stored into useful electrical energy (inside the satellite) is given by the First Law of Thermodynamics:

$$P = Q_H - Q_L.$$

The goal is to maximize P. However, the Second Law of Thermodynamics provides an additional constraint, which is that the efficiency of the satellite cannot exceed the Carnot efficiency:

$$\frac{P}{Q_H} \leq 1 - \frac{T_L}{T_H}$$

Moreover, the Third Law of Thermodynamics implies $T_L > 0$.

Now, we can maximize P subject to the above constraints. Note that the above inequality is saturated when P is maximal. Suppose it was not. Then, we can decrease T_L a little bit (since it is positive), which will cause Q_L to decrease a little bit. Keeping everything else fixed, this will cause $P = Q_H - Q_L$ to increase a little bit. As long as the changes were small enough, the inequality will still hold. Thus, the inequality must be saturated when P is maximal. In other words, we have $\frac{Q_L}{T_L} = \frac{Q_H}{T_H}$

Also, note that $\epsilon_H = 1$ when P is maximal. Suppose otherwise. Then ϵ_H can be increased a little bit, and T_H increased until $\frac{Q_H}{T_H}$ is equal to its original value of $\frac{Q_L}{T_L}$. Note that Q_H has increased since T_H increased while keeping $\frac{Q_H}{T_H}$ constant. Hence, fixing Q_L and T_L, this would cause $P = Q_H - Q_L$ to strictly increase. Thus, $\epsilon_H = 1$ when P is maximal.

Finally, note that $\epsilon_L = 1$ when P is maximal. Suppose otherwise. Then ϵ_L can be increased a little bit, and T_L be decreased a little bit, such that $\frac{Q_L}{T_L} = \epsilon_L \sigma A T_L^3$ remains fixed.

Note that Q_L has decreased since T_L decreased while keeping $\frac{Q_L}{T_L}$ constant. Hence, fixing everything else, we see that $P = Q_H - Q_L$ would strictly increase. Thus, we must have $\epsilon_L = 1$ when P is maximal.

This simplifies our problem to the following:

$$\begin{aligned} \text{maximize} \quad & P = Q_H - Q_L && (1) \\ \text{subject to} \quad & Q_H = AS - \sigma A T_H^4 && (2) \\ & Q_L = \sigma A T_L^4 && (3) \\ & Q_L \frac{T_H}{T_L} = Q_H. && (4) \end{aligned}$$

Define $r_L^4 = \frac{\sigma T_L^4}{S}$ and $r_H^4 = \frac{\sigma T_H^4}{S}$. Then substituting (2) and (3) into (1) and (4) gives us:

$$\begin{aligned} \text{maximize} \quad & P = AS(1 - r_H^4 - r_L^4) && (1') \\ \text{subject to} \quad & r_L^3 r_H = 1 - r_H^4. && (4') \end{aligned}$$

Now,

$$1 - r_H^4 - r_L^4 = r_L^3 r_H - r_L^4 = \frac{r_L^3 r_H - r_L^4}{r_L^3 r_H + r_H^4} = \frac{x^3 - x^4}{x^3 + 1},$$

where $x := \frac{r_L}{r_H}$. This expression is maximized for $x > 0$ at $x = 0.6925$, with maximum value 0.07666.

Thus, the maximum value of P is $0.07666AS$.