# Solution to problem 5 (Satellite)

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## Definitions

Let $\epsilon_1$ and $\epsilon_2$ be the emissivity of the paints on the faces with the incident light (Face 1) and the opposite face (Face 2) respectively. Inside the satellite there is a set of devices that we will call for short as "machine" that transforms thermal energy into electrical energy with an efficiency $\eta$ extracting energy from face 1 with temperature $T_1$[1] and giving energy to face 2 with temperature $T_2$. Let $Q_1$ be the heat delivered by face 1 to the machine and $Q_2$ be the heat released by the machine.

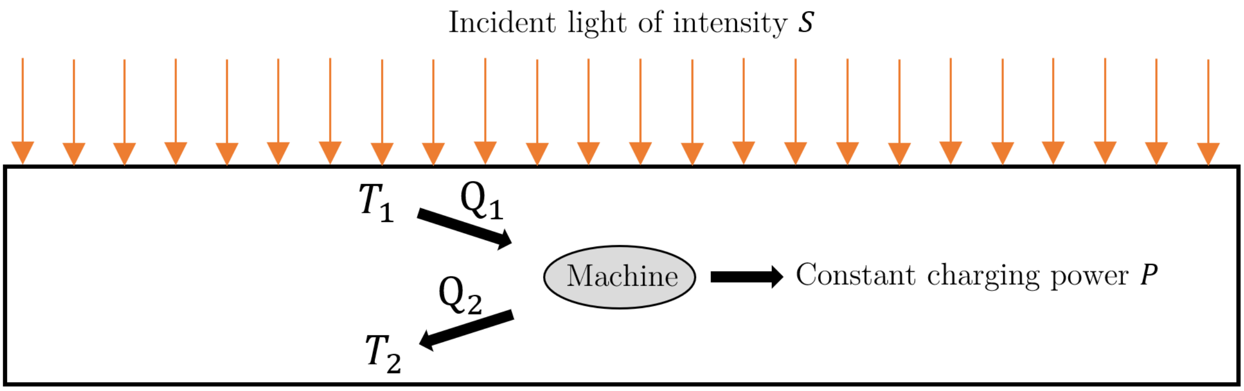


Figure 1: Structure

Since $\sqrt{A} \gg d$, we will neglect the emissive effects of the edge.

## Solution

By the first law of thermodynamics and the Stefan-Boltzmann law we get

$$Q_1 = \epsilon_1 A(S - \sigma T_1^4)$$

$$Q_2 = \epsilon_2 A\sigma T_2^4$$

The efficiency $\eta$ cannot be higher than that of the Carnot cycle, in order to obtain the highest charging power, we require the highest efficiency, therefore the theoretical limit is given when

$$\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{Q_2}{Q_1}$$

$$\frac{T_2}{T_1} = \frac{Q_2}{Q_1} \tag{1}$$

[1] This is because any machine can only extract energy from deposits at a lower temperature than the faces, with the aim that the load power is maximum, the limit values will be taken

This can be obtained by analyzing the entropy

$$\Delta S_{universe} = -\frac{Q_1}{T_1} + \frac{Q_2}{T_2}$$

Note that if $P > 0$, then $Q_1 > Q_2$, but $\Delta S_{universe} \geq 0$ implying that $Q_2 \frac{T_1}{T_2} \geq Q_1$ from where it is mandatory that $T_1 \geq T_2$ [2]. Therefore, if the process is reversible, equation (1) is obtained, continuing

$$\frac{\epsilon_2 A \sigma T_2^4}{\epsilon_1 A(S - \sigma T_1^4)} = \frac{T_2}{T_1}$$

$$T_2 = \sqrt[3]{\frac{\epsilon_1}{\epsilon_2}\left(\frac{S}{\sigma T_1} - T_1^3\right)} \Rightarrow \eta = 1 - \sqrt[3]{\frac{\epsilon_1}{\epsilon_2}\left(\frac{S - \sigma T_1^4}{\sigma T_1^4}\right)}$$

Finally the charging power will be

$$P = \eta Q_1$$

$$P = \epsilon_1 A(S - \sigma T_1^4)\left(1 - \sqrt[3]{\frac{\epsilon_1}{\epsilon_2}\left(\frac{S - \sigma T_1^4}{\sigma T_1^4}\right)}\right)$$

Defining the dimensionless coefficient $x = \frac{S - \sigma T_1^4}{S}$ we are left with

$$P = SA\epsilon_1 x\left(1 - \sqrt[3]{\frac{\epsilon_1}{\epsilon_2}\frac{x}{(1-x)}}\right)$$

It is easy to see that the emissivity $\epsilon_2 = 1$ minimizes the value it is subtracting, thus maximizing $P$. Having $\epsilon_1 \leq 1$ is equivalent to having an area painted black $A^*$ and everything else white, so $\epsilon_1 = A^*/A$, wasting energy that could be spent on increasing temperature $T_1$ and improving efficiency, so to maximize charging power $\epsilon_1 = 1$ is required . then we are left

$$\frac{P}{SA} = x\left(1 - \sqrt[3]{\frac{x}{1-x}}\right)$$

The maximum is reached when the derivative is 0, that is

$$0 = \left(1 - \sqrt[3]{\frac{x}{1-x}}\right) - \frac{x}{3}\left(\frac{x}{1-x}\right)^{-\frac{2}{3}}\left(\frac{1-x+x}{(1-x)^2}\right) = 1 - \left(\frac{x}{1-x}\right)^{\frac{1}{3}} - \frac{x^{\frac{1}{3}}}{3(1-x)^{\frac{4}{3}}}$$

$$\left(\frac{x}{1-x}\right)^{\frac{1}{3}}\left(1 + \frac{1}{3(1-x)}\right) = 1 \Rightarrow p(x) = x\left(\frac{4}{3} - x\right)^3 - (1-x)^4 = 0$$

[2] This tells us that the reverse process, taking heat from Face 2 and expelling it to Face 1, cannot exist or needs $P < 0$.

## Numerical calculation

Being a complex equation, we will solve it by numerical calculation, note that $x$ is limited between 0 and 1. Let's take numerical values in steps:

| $x$ | $p(x)$ |
|---|---|
| 0 | -1 |
| 0.1 | -0.468496296 |
| 0.2 | -0.118459259 |
| 0.3 | 0.090911111 |
| 0.4 | 0.195614815 |
| 0.5 | 0.226851852 |
| 0.6 | 0.211022222 |
| 0.7 | 0.169725926 |
| 0.8 | 0.119762963 |
| 0.9 | 0.073133333 |
| 1.0 | 0.037037037 |

From there we can see that it will be between 0.2 and 0.3, again

| $x$ | $p(x)$ |
|---|---|
| 0.20 | -0.118459259 |
| 0.21 | -0.091823842 |
| 0.22 | -0.066552971 |
| 0.23 | -0.042608968 |
| 0.24 | -0.019954631 |
| 0.25 | 0.001446759 |
| 0.26 | 0.021631443 |
| 0.27 | 0.040635180 |
| 0.28 | 0.058493250 |
| 0.29 | 0.075240454 |
| 0.30 | 0.090911111 |

From where it is seen that the root is between 0.24 and 0.25, we repeat the process

| $x$ | $p(x)$ |
|---|---|
| 0.240 | -0.019954631 |
| 0.241 | -0.017758711 |
| 0.242 | -0.015575283 |
| 0.243 | -0.013404312 |
| 0.244 | -0.011245761 |
| 0.245 | -0.009099593 |
| 0.246 | -0.006965773 |
| 0.247 | -0.004844264 |
| 0.248 | -0.002735029 |
| 0.249 | -0.000638033 |
| 0.250 | 0.001446759 |

From where it is seen that the root is between 0.249 and 0.250, we repeat the same thing but this time with the value of $P/SA$.

| $x$ | $p(x)$ | $P/SA$ |
|---|---|---|
| 0.2490 | -0.0006380330 | 0.076660107 |
| 0.2491 | -0.0004290050 | 0.076660163 |
| 0.2492 | -0.0002201000 | 0.076660197 |
| 0.2493 | -0.0000113160 | 0.076660209 |
| 0.2494 | 0.0001973457 | 0.076660200 |
| 0.2495 | 0.0004058858 | 0.076660168 |
| 0.2496 | 0.0006143040 | 0.076660114 |
| 0.2497 | 0.0008226004 | 0.076660039 |
| 0.2498 | 0.0010307751 | 0.076659941 |
| 0.2499 | 0.0012388280 | 0.076659822 |
| 0.2500 | 0.0014467592 | 0.076659681 |

Of the latter, it will be between 0.2493 and 0.2494, we already have enough precision to ensure at least 4 significant figures, which are

$$P_{max} = 0.07666SA$$