Physics Cup 2023 – Problem 1

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A solid cylinder of radius R and height H has density ρ_c and is immersed in water of density ρ_w. The cylinder is initially kept at rest so that its axis is vertical, and the distance between its bottom face and the bottom of the water container is h. The water container has a flat rigid bottom, and the depth of water in it is larger than $H+h$. At a certain moment, the cylinder is released and starts falling. How long it will take for the cylinder to hit the bottom? Neglect viscosity. Assume that $\rho_c > \rho_w$, $H < R$ and $10\rho_c h \ll \rho_w R$.

Solution

One might be attempted to take the water into account using hydrodynamical results for infinite flow, but this won't be a good idea in this case – the last assumption hints that the flow of water between the cylinder and the bottom must be much more interesting then that, because of the bottom.

Here's a rough model. Water flows out in radially dependent velocity $v(r)$ up until $r = R$, and then just goes everywhere (up?), taking virtually no kinetic energy (as there is a very large volume of water moving, and keeping the current mv constant, higher mass is lower energy).

Suppose the height changes at rate $\frac{dh}{dt}$. At radius r, we construct the continuity equation. Between r and $r+dr$ we have influx of $2\pi rhv(r)\,dt$ and an outflux of $2\pi(r+dr)\,hv(r)\,dt$. However, water being incompressible, the volume of water there must decrease by $2\pi r dr dh$.

This can be simplified. Look on the whole circle (instead of the infinitesimally small ring) – its influx is $\pi r^2 dh$ and its outflux is $2\pi rhvdt$. We get

$$\pi r^2 dh = 2\pi rhvdt \implies v(r) = \frac{r}{2h}\frac{dh}{dt}$$

The energy of this is given by integration on tiny rings:

$$\int_0^R \frac{1}{2}\underbrace{2\pi rhdr}_{dV}\rho_w\left(\frac{r}{2h}\frac{dh}{dt}\right)^2 = \frac{\pi}{4h}\rho_w\left(\frac{dh}{dt}\right)^2\int_0^R r^3 dr = \frac{\pi R^4}{16h}\rho_w\left(\frac{dh}{dt}\right)^2$$

and this gives us an "effective mass"; this effective mass goes up with h getting smaller!

The actual mass of the cylinder leads to kinetic energy $\rho_c\pi R^2 H\left(\frac{dh}{dt}\right)^2$. The ratio between the two is $\frac{E_k^c}{E_k^2} = \frac{\rho_c\pi R^2 H}{\rho_w\frac{\pi R^4}{16h}} = \frac{16\rho_c hH}{\rho_w R^2}$. We know that $\frac{\rho_c h}{\rho_w R} \ll \frac{1}{10}$, and multiply by $\frac{H}{R} < 1$ to conclude that $\frac{E_k^c}{E_k^w} \ll 1$. Added mass can be neglected as well, as it won't be larger than the cylinder's mass (not with $\rho_w < \rho_c$).

Hence, we have a falling body with an effective mass. The gravity along with buoyancy give a force of $(\rho_c - \rho_w)\,\pi R^2 Hg$ down. Mind you, g isn't explicitly given, but it's the only reason the cylinder will start falling, so we shall assume it exists as well.

The acceleration is $\frac{F_{tot}}{M_{eff}}$ – this can be observed from energy, or just a plain observation from the structure of the force and the water kinetic energy. The acceleration is, therefore, given by

$$a = \frac{(\rho_c - \rho_w)\,\pi R^2 Hg}{\frac{\pi R^4}{16h}\rho_w} = \frac{16h\,(\rho_c - \rho_w)\,Hg}{R^2\rho_w} = 16\frac{\rho_c - \rho_w}{\rho_w}\frac{Hg}{R^2}h.$$

That's harmonic! we call the coefficient of h ω^2 (signs are all over the place anyway), and get the final result:

$$\omega = 4\sqrt{\frac{\rho_c - \rho_w}{\rho_w}}\sqrt{\frac{Hg}{R^2}} \implies t_{fall} = \frac{T}{4} = \frac{\pi}{2\omega} = \frac{\pi}{8}\sqrt{\frac{\rho_w}{\rho_c - \rho_w}}\sqrt{\frac{R^2}{Hg}}$$

As sanity check, note that vanishing $\rho_c - \rho_w$ takes us to infinity (no gravity), infinite ρ_c, g or H leads us to zero (very large down force), and larger R does the opposite (large "waterbag", also required for units). Mind the h independence (for low enough h)!

Oops, not exactly...

Changing mass isn't that simple. The energy we have is

$$\frac{\pi R^4}{16}\frac{1}{h}\left(\frac{dh}{dt}\right)^2 + (\rho_c - \rho_w)\,\pi R^2 gh$$

Deriving this will give us a not-so-trivial force equation! Denote $\omega^2 = 16\frac{\rho_c-\rho_w}{\rho_w}\frac{Hg}{R^2}$ to get

$$\frac{2h\dot{h}\ddot{h} - \dot{h}^3}{h^2} + \omega^2\dot{h} = 0$$

$$2h\ddot{h} - \dot{h}^2 + \omega^2 h^2 = 0$$

This isn't trivial. But recognize a familiar pattern and substitute $g = \sqrt{h}$, $h = g^2$, $\dot{h} = 2g\dot{g}$, $\ddot{h} = 2\dot{g}^2 + 2g\ddot{g}$, to get

$$2g^2\left(2\dot{g}^2 + 2g\ddot{g}\right) - (2g\dot{g})^2 + \omega^2 g^4 = 0$$

$$4\dot{g}^2 + 4g\ddot{g} - 4\dot{g}^2 + \omega^2 g^2 = 0$$

$$4\ddot{g} + \omega^2 g = 0$$

and we get g that's a cosine or a sine with $\frac{\omega}{2}$ – a cosine, as we start in rest. However, that means h is $\cos^2$. So h is $A\cos\left(\frac{\omega t}{2}\right)^2 = \frac{A}{2}\left(\cos(\omega t) + 1\right)$. That means we don't wait for a quarter-period, but rather a half-period, maximum to minimum!

$$t_{fall} = \frac{T}{2} = \frac{\pi}{\omega} = \frac{\pi}{4}\sqrt{\frac{\rho_w}{\rho_c - \rho_w}}\sqrt{\frac{R^2}{Hg}}$$