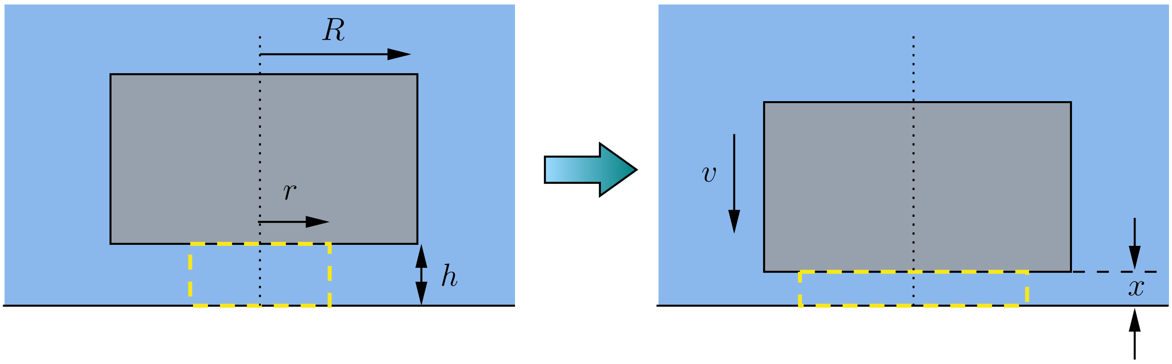
Physics Cup 2023 Problem 1



Let the distance from the bottom of the cylinder to the bottom of the container be $x \leq h$, which decreases when the cylinder sinks. Let the speed of the cylinder be $v = -\dot{x}$.

Consider a cylinder of radius r of water coaxial to the cylinder, shown by the yellow rectangle in the figure above. The volume of water inside the cylinder is $V = \pi r^2 x$. As the cylinder is falls down (i.e. x decreases), V decreases so the water is pushed out. Noting that water is **incompressible**, we have

$$2\pi r x \frac{\mathrm{d}r}{\mathrm{d}t} + \pi r^2 \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{rv}{2x}$$

This is the radial velocity (pointing outwards) of the water. As the cylinder falls down, part of its kinetic energy is converted into the kinetic energy of the water as it pushes them out. The kinetic energy of water (we ignore the water which is not beneath the cylinder as they only contribute negligible amount of kinetic energy) due to the radial velocity is

$$E_r = \frac{1}{2}\rho_w \int_0^R \left(\frac{rv}{2x}\right)^2 (2\pi r x)\,\mathrm{d}r = \frac{\pi \rho_w R^4 v^2}{16x}$$

Note that the water will also have components of velocity parallel to the cylinder axis. We can estimate the order of the associated kinetic energy $E_{||}$, which is much less than E_r, so we can ignore it when we consider energy conservation later:

$$E_{||} \sim \frac{1}{2}\rho_w \int_0^R 2\pi r x v^2\,\mathrm{d}r = \frac{\pi}{2} x \rho_w R^2 v^2 \ll E_r$$

In the system, the potential energy of the cylinder is transformed into the potential energy of the water, the kinetic energy of the cylinder and the kinetic energy of the water, so we have:

$$(\rho_c - \rho_w)\pi R^2 Hg(h-x) = \frac{1}{2}\rho_c \pi R^2 H v^2 + \frac{\pi \rho_w R^4 v^2}{16x}$$

$$v^2 = \frac{16(\rho_c - \rho_w)gHx(h-x)}{8\rho_c Hx + \rho_w R^2}$$

Note that $8\rho_c Hx \leq 8\rho_c Hh < 10\rho_c Hh < 10\rho_c Rh \ll \rho_w R^2$ as implied from the condition given in the question that $10\rho_c h \ll \rho_w R$, hence the second term in the denominator is much greater than the first term. Notice that the velocity v tends to zero when x tends to zero, so it shows that the water acts as a cushion to stop the fall of the cylinder. We can then integrate to find the time T, by applying taylor expansion of the numerator of the integrand:

$$\begin{aligned}
T &= \int_0^T \mathrm{d}t \\
&= \int_0^h \frac{1}{v}\,\mathrm{d}x \\
&= \int_0^h \sqrt{\frac{\rho_w}{\rho_c - \rho_w}\frac{R^2}{16gH}}\sqrt{\frac{1 + 8\rho_c Hx/\rho_w R^2}{x(h-x)}}\,\mathrm{d}x \\
&\approx \int_0^h \sqrt{\frac{\rho_w}{\rho_c - \rho_w}\frac{R^2}{16gH}}\frac{1 + 4\rho_c Hx/\rho_w R^2}{\sqrt{x(h-x)}}\,\mathrm{d}x \\
&= \boxed{\sqrt{\frac{\rho_w}{\rho_c - \rho_w}\frac{R^2}{gH}}\frac{\pi}{4}\left(1 + \frac{2\rho_c Hh}{\rho_w R^2}\right)}
\end{aligned}$$

Here, the integral $\int_0^a \frac{1+bx}{\sqrt{x(a-x)}} = \pi\left(1 + \frac{ab}{2}\right)$ is used. We can prove by splitting the integral into two as follows:

$$\int_0^a \frac{1}{\sqrt{x(a-x)}}\,\mathrm{d}x = \int_0^a \frac{1}{\sqrt{(a/2)^2 - (x-a/2)^2}}\,\mathrm{d}x = \pi$$

$$\int_0^a \frac{bx}{\sqrt{x(a-x)}}\,\mathrm{d}x = \frac{1}{2}\left[\int_0^a \frac{bx}{\sqrt{x(a-x)}}\,\mathrm{d}x + \int_0^a \frac{b(a-x)}{\sqrt{x(a-x)}}\,\mathrm{d}x\right] = \frac{1}{2}\int_0^a \frac{ab}{\sqrt{x(a-x)}}\,\mathrm{d}x = \frac{1}{2}ab\pi$$