Physics Cup 2023, Problem 1

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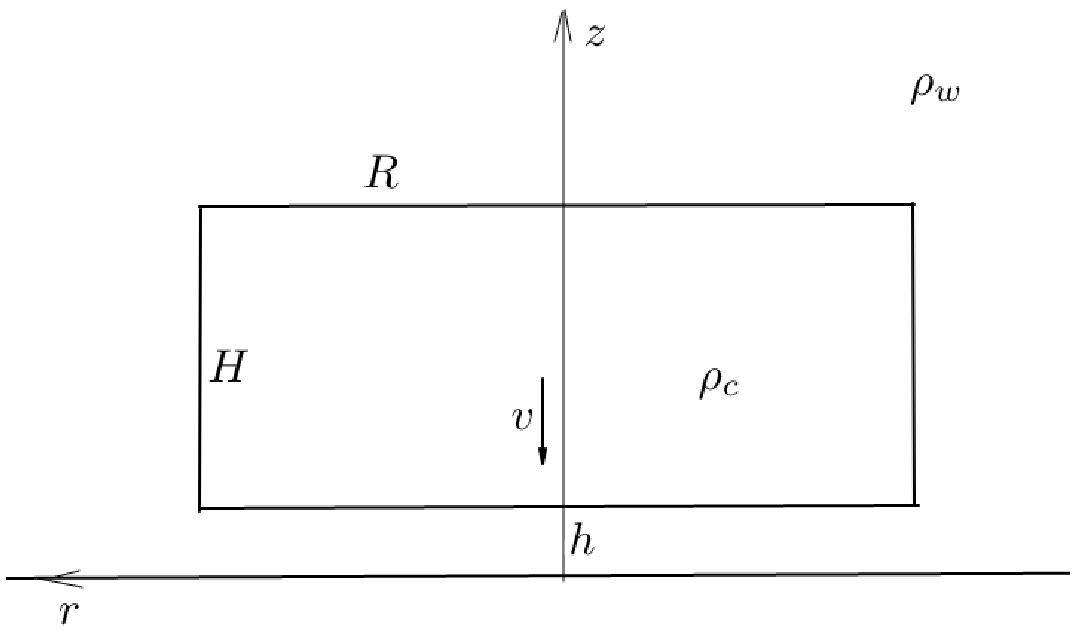

Figure 1: Problem setup

1 Assumptions

$$\rho_c > \rho_w \tag{1}$$

$$R > H \tag{2}$$

$$10\rho_c h \ll \rho_w R \tag{3}$$

2 Initial remarks

1. Water flow velocity directly above the ground is tangential to it because $\hat{n}\cdot\vec{u} = 0$, where $\vec{u}$ denotes the velocity of a water element at a position $\vec{r}$ and time t: $\vec{u} \equiv \vec{u}(\vec{r}, t)$. Likewise, in cylinder's reference frame water can only have velocity tangential to the cylinder's surface. Meaning that in the lab frame fluid elements in contact with the cylinder's top or bottom have to have a velocity component in the $\hat{z}$ direction equal to cylinder's velocity $\vec{v}$ which is always in the $-\hat{z}$ direction. $\vec{v} = -v\hat{z}$.

2. **The fluid flow is at all times irrotational.** This holds because the flow is irrotational at $t = 0$ (there is no flow), and afterwards, the only forces acting on the fluid are gravitational and normal forces comming from the ground and cylinder. There are no shear forces because of the assumption of inviscid flow. Therefore, we are dealing with a potential flow meaning there exists a potential φ which can be used to determine the velocity at every point in the fluid according to $\vec{u} = \nabla\varphi$. A corrolary of this fact is the existence of equipotential surfaces which fluid streamlines intersect perpandicularly. Also, there aren't any singularities in the potential.

3. The fluid in question is water so **incompressibility** is assumed and the continuity equation is expressed in the form $\oint_A \vec{u} \cdot d\vec{A} = 0$
4. Another consequence of no viscosity is the **conservation of energy** of the system water-cylinider.
5. The problem exhibits cylindrical symmetry so the fluid velocity can be expressed as a function in the distance from the cylinder's axis r, the vertical coordinate z and time t.
6. In this solution the separation between the cylinder and the ground h is taken as a function of time: $h \equiv h(t)$ and the initial separation is denoted h_0. Assumption 3 holds for all h since it is decreasing from h_0 to 0.

3 Kinetic energy of potential flows

In this section we relate the kinetic energy of a fluid volume V contained in between two equipotential surfaces and the fluid flux Φ through them. Let these two surfaces be infinitesimally separated. Kinetic energy of the volume of fluid between these surfaces is then:

$$dK = \frac{1}{2}\rho_w dw \int_A u^2 dA \tag{4}$$

where dw is the distance between them A is their area and u the magnitude of velocity at each point along the surfaces which is essentialy the same at each surface. Furthermore, because $\vec{u}$ is perpandicular to the surfaces we can express the flux through them as

$$\Phi = \int_A u dA \tag{5}$$

and define an average velocity $\bar{u}$

$$\bar{u} = \frac{1}{A}\int_A u dA = \frac{\Phi}{A} \tag{6}$$

But the average of the square $\bar{u^2}$ over the surface A can be expressed as

$$\bar{u^2} = \frac{1}{A}\int_A u^2 dA = \bar{u}^2 + \sigma^2 = \left(\frac{\Phi}{A}\right)^2 + \sigma^2 \tag{7}$$

where we define variance σ^2 as

$$\sigma^2 = \frac{1}{A}\int_A (u - \bar{u})^2 dA \tag{8}$$

Therefore, the kinetic energy dK is

$$dK = \frac{1}{2}\rho_w dV \left(\left(\frac{\Phi}{A}\right)^2 + \sigma^2\right) \tag{9}$$

Meaning that $dK \geq \frac{1}{2}\rho_w dV(\frac{\Phi}{A})^2$ – the more "unhomogenous" the velocity distribution over the surface A is, the more kinetic energy V contains while we keep the flux Φ and equipotential surfaces the same.
This opens the possibility of comparing the kinetic energy in different parts of the fluid (although the exact form of the velocity field remains unknown) by comparing an alternative imaginary velocity flows with larger σ^2 than the actual one.

4 Fluid motion right below the cylinder

Let's observe what happens when cylinder drops by an infinitessimal height $ds = -dh$. Volume of water displaced by the cylinder is $dsR^2\pi$. This water exits the cylinder of water under the cylinder itself through its sides of area $S = 2R\pi h$. Equation of continuity tells us that

$$2R\pi h u_{S,avg} = \frac{dsR^2\pi}{dt} = vR^2\pi \tag{10}$$

$$u_{S,avg} = \frac{R}{2h}v \tag{11}$$

where $u_{S,avg}$ is defined as

$$u_{S,avg} = \frac{1}{2R\pi h}\int_S \vec{u} \cdot d\vec{S} \tag{12}$$

We assume a monotonously decreasing function of the $\hat{z}$ component of water velocity u_z from $u_z(z=0)=0$ to $u_z(z=h)=-v$ for all points on the surface S according to remark 1: $u_z \in \langle -v, 0\rangle$ We have assumption 3 which along with assumption 1 implies $h \ll R$ Therefore,

$$u_{S,avg} \gg v \tag{13}$$

And since the component of flow velocity through the surface S paralel to it is negligible compared to the velocity component perpandicular to it, we take the magnitude of velocity to be approximately $u_{S,avg}$. (It is implausible that u(R) depends on z[1])

$$u(R) = u_{S,avg} = \frac{R}{2h} v \tag{14}$$

The same argument can be used for all coaxial cylindrical surfaces of some radius $r < R$ as long as $r \gg h$. But because $R \gg 10h$ (due to assumptions 3 and 1) $r \gg h$ mostly holds.

$$\Rightarrow u(r) = \frac{r}{2h} v \tag{15}$$

In the calculation of kinetic energy contained it the volume of water below the cylinder in any given moment we integrate over volume elements $dV = 2\pi h r dr$. Meaning that the integral $\int u^2 dV \propto r^4$ over those elements for which equation 15 doesn't hold (close to the axis where r is of order $h \ll R$), so the kinetic energy of those elements contributes insignificantly:

$$\begin{aligned} K_{bc} &= \frac{1}{2}\rho_w \int_0^R u^2 dV \\ &= \frac{1}{2}\rho_w \int_0^R \frac{r^2 v^2}{4h^2} 2\pi h r dr \\ &= \frac{\pi \rho_w v^2}{4h} \int_0^R r^3 dr \end{aligned}$$

$$K_{bc} = \frac{\pi \rho_w v^2 R^4}{16h} \tag{16}$$

5 Fluid motion elsewhere

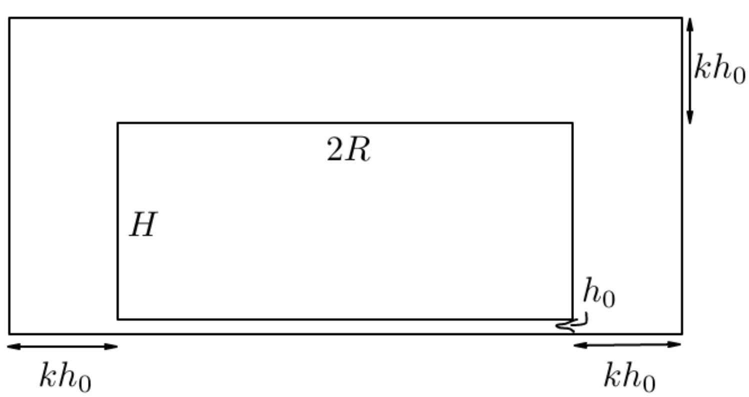

Figure 2: Imaginary confinement of the flow

It is difficult to determine the exact form of the velocity field $\vec{u}$ outside of the thin region below the cylinder or determine the kinetic energy contained in the rest of the fluid, but we may calculate approximate values for an

[1]A possible (if a little hand-wavy) justification for this is that potential flows for given boundary conditions have solutions of minimum energy (Kelvin). A solution with an unecessarily wide distribution of velocities necessarily has larger kinetic energy because it grows with u^2 and thus is not a candidate for an actual solution.

alternative velocity field which clearly has much larger kinetic energy according to the conclusions of section 3 and show that it is still less then the kinetic energy contained below the cylinder.
Let's imagine that the water is contained within a cylinder of radius $R + kh_0$ and height $h_0 + H + kh_0$ where k is some dimensionless constant (figure 2). k is taken such that

$$kh_0\rho_c \ll R\rho_w \tag{17}$$

holds which means that k is of order 10 (assumption 3). This gives the water only a thin region along the cylinder along which it can travel but still a lot wider than the region below it.
We estimate the kinetic energy of water to the sides and above the cylinder. To the sides we take the water to flow with speed u_{side}. According to the continuity condition:

$$u_{side}\pi\left((R + kh_0)^2 - R^2)\right) = \Phi = R^2\pi v \tag{18}$$

$$u_{side} = \frac{vR^2}{kh_0(kh_0 + 2R)} \tag{19}$$

The corresponding kinetic energy:

$$\begin{aligned} K_{side} &= \frac{1}{2}\rho_w u_{side}^2 \left((R + kh_0)^2 - R^2)\right)\pi H \\ &= \frac{\pi\rho_w v^2 R^4 H}{2kh_0(kh_0 + 2R)} \\ &\approx \frac{\pi\rho_w v^2 R^3 H}{4kh_0} \\ &< \frac{\pi\rho_w v^2 R^4}{4kh_0} \end{aligned} \tag{20}$$

where the last inequality is due to assumption 2 and the approximation due to eq.17

$$\Rightarrow K_{side} < \frac{4}{k}K_{bc} \tag{21}$$

The region of water above the cylinder is analoguous to the one below it except its heigth is between kh_0 and $(k+1)h_0$ but since eq.17 holds, analogous conclusions to those from section 4 hold because the only real difference is the direction of the flow.

$$\begin{aligned} \Rightarrow K_{above} &= \frac{\pi\rho_w v^2 R^4}{16((k + 1)h_0 - h)} \\ &= \frac{K_{bc}h}{(k + 1)h_0 - h)} = \frac{K_{bc}}{(k + 1)\frac{h_0}{h} - 1)} \\ &\leq \frac{K_{bc}}{k} \end{aligned} \tag{22}$$

We can ignore the energy contained in the regions connecting the mentioned regions of flow because their volume is negligible compared to the volume above and/or below the cylinder and their flow velocity cannot be much higher than the maximum velocity in other regions. Thus the total kinetic energy in regions not below the cylinder:

$$K_{side} + K_{above} < \frac{5}{k}K_{bc} \tag{23}$$

This kinetic energy is less than K_{bc} because k is of order 10, but the actual kinetic energy of the actual flow has to be much smaller than this because of the conclusions of section 3 because our flow clearly has large variance σ^2 as it was defined there as the velocity is 0 outside of our imaginary container and has to be much larger inside of it to preserve the flux Φ. It is reasonable assume a much more spread out flow in reality.
An objection that might be raised is that there is no guarantee that this imaginary flow has equipotential surfaces which are the same as actual ones in this confined region. In the region where water flows upwards along the cylinder we *can* actually assume this because close to the cylinder the flow must be paralel to it an thus the equipotential surfaces have to be perpandicular to the sidewall of the cylinder just as in our imaginary flow. The flow above in reality presumably has velocities mostly in $-\hat{z}$ direction and of magnitude of order v (remark 1) because nothing

compells it to accelerate to the degree to wich it must be below the cylinder. This means that its kinetic energy is much less than in our imaginary flow anyway.
All of this serves to justify neglecting the water's kinetic energy in regions other than directly below the sinking cylinder.

$$K_{\neg bc} \ll K_{bc} \tag{24}$$

6 Conservation of energy

We can thus express the conservation of energy as:

$$K_{bc} + K_c + \Delta U_c + \Delta U_w = 0 \tag{25}$$

where K_c is cylinder's kinetic energy, ΔU_c is the change in cylinder's gravitational potential energy when as it falls a distance $-\Delta h = h_0 - h$ and ΔU_w is the corresponding change in the potential energy of the water. For K_c we have simply:

$$K_c = \frac{1}{2}\rho_c R^2 \pi H v^2 \tag{26}$$

And for the potential energies:

$$\begin{aligned} \Delta U_c + \Delta U_w &= \rho_c R^2 \pi H g \Delta h + \rho_w R^2 \pi \Delta h g(-H) \\ &= \pi R^2 H g(\rho_c - \rho_w)(h - h_0) \end{aligned} \tag{27}$$

Combinig equations 25, 26, 27, 16 and noticing $v = -\dot{h}$ yields:

$$\frac{\pi \rho_w \dot{h}^2 R^4}{16h} + \frac{1}{2}\rho_c R^2 \pi H \dot{h}^2 + \pi R^2 H g(\rho_c - \rho_w)(h - h_0) = 0 \tag{28}$$

We introduce the following substitutions to simplify calculation:

$$\begin{aligned} A &= \frac{\pi}{16}\rho_w R^4 \\ B &= \frac{1}{2}\rho_c R^2 \pi H \\ C &= \pi R^2 H g(\rho_c - \rho_w) \\ D &= -C h_0 \end{aligned} \tag{29}$$

Thus,

$$A\frac{\dot{h}^2}{h} + B\dot{h}^2 + Ch + D = 0 \tag{30}$$

$$\Rightarrow \dot{h} = -\sqrt{\frac{-Ch - D}{\frac{A}{h} + B}} \tag{31}$$

$$\Rightarrow \int_{h_0}^{0} \frac{-dh}{\sqrt{\frac{-Ch^2 - Dh}{A\left(1 + \frac{Bh}{A}\right)}}} = \int_0^T dt \tag{32}$$

According to assumptions 2 and 3:

$$\frac{Bh}{A} = \frac{8\rho_c H h}{\rho_w R^2} < \frac{8\rho_c h}{\rho_w R} \ll 1 \tag{33}$$

So, the above integral can be simplified:

$$\begin{aligned} T &= \int_{h_0}^{0} \frac{-dh}{\sqrt{\frac{-Ch^2 - Dh}{A}}} \\ &= \sqrt{A}\frac{2\arcsin\sqrt{\frac{Ch}{-D}}\Big|_0^{h_0}}{\sqrt{C}} \\ &= \pi\sqrt{\frac{A}{C}} \end{aligned} \tag{34}$$

Finally:

$$T = \frac{\pi}{4}\sqrt{\frac{R^2}{(\frac{\rho_c}{\rho_w} - 1)Hg}} \tag{35}$$

Interestingly the total sink time for the cylinder T doesn't depend on h_0 in the limit of assumption 3.