Problem 1 - Physics Cup 2023

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1 Outline of solution

First of all, we'll invoke conservation of energy in order to establish a relation between the distance between the bottom of the cylinder and the bottom of the container and its velocity, gradually excluding all the terms that can be shown to be negligible. We'll then proceed to integrate the equation obtained in order to find the time needed T.

2 Preliminary assumptions

Since the fluid in non-viscous, there's no energy dissipation due to the movement of the fluid. Moreover, since the system exhibits cylindrical symmetry and the cylinder is released from rest when its axis is vertical, its faces will always be parallel to the bottom of the container and there won't be any rotation around its axis.

3 Calculating the total energy

Let $x(t)$ be the distance of the bottom face of the cylinder from the bottom of the container. We can write the total energy of the system as

$$E = K_c + U_c + K_w + U_w$$

where K_c, K_w represent the cylinder and the fluid's kinetic energy and U_c, U_w represent their potential energy. We'll begin by writing down the cylinder's kinetic term. Since its motion is purely translational, we get

$$K_c = \frac{1}{2}\pi R^2 H \rho_c \dot{x}^2$$

As for its potential energy, setting $U_c = 0$ for $x = 0$ (we're free to do this, since potential energy is defined only up to a constant term) gives

$$U_c = \pi R^2 H \rho_c g x$$

Let's now focus on the fluid. If its potential energy is U_0 when the cylinder is released, its energy at an arbitrary $x(t)$ will be

$$U_w = U_0 + \pi R^2 H \rho_w g(h - x)$$

In fact, when the cylinder moves down by a distance $h - x$, the portion of water of volume $\pi R^2(h - x)$ will be pushed away in the surrounding space, and an equal portion of water will occupy the same volume of space above the cylinder, which had been left empty. The potential energy of the remaining fluid remains unaltered and, since the cylinder's height is H, we'll have

$$\Delta U_w = \pi R^2 H \rho_w g(h - x)$$

from which the aforementioned formula follows. By setting

$$U_0 = -\pi R^2 H \rho_w g h$$

(we can do this for the same reasons explained before) we get

$$U_w = -\pi R^2 H \rho_w g x$$

Finding its kinetic energy is trickier. We begin by calculating the contribution from the fluid directly below the cylinder (i.e. within a distance R from its axis, in the region below the bottom face). Since water is incompressible and its density can be considered uniform, the velocity field satisfies the integral continuity equation:

$$\oint_{\partial V} \rho_w \vec{v} \cdot \mathrm{d}\vec{A} = -\dot{m}_V$$

where m_V is the total mass of water contained in the volume V. Consider a fictitious cylinder of radius r with axis coincident with that of the solid cylinder, bottom base touching the floor and top base at an height $x(t)$. Applying the continuity equation to its volume yields

$$\int_0^{x(t)} 2\pi r \rho_w v_r(x', r) \mathrm{d}x' = -\pi r^2 \rho_w \dot{x} \Rightarrow \int_0^{x(t)} v_r(x', r) \mathrm{d}x' = -\frac{1}{2} r \dot{x}$$

This means that the radial velocity of the fluid is in the range of $\frac{R}{x}\dot{x}$; on the other hand, its vertical velocity can be estimated to be of the same order of magnitude of $\dot{x}$. By our assumptions

$$\frac{x}{R} < \frac{h}{R} \ll \frac{\rho_w}{10\rho_c} < \frac{1}{10}$$

therefore the vertical motion of the fluid below the (real) cylinder can be neglected. The motion of water can be considered to be approximately vortex free in that region (since we can treat the fluid as ideal), hence

$$\oint \vec{v} \cdot \mathrm{d}\vec{l} = 0$$

around every closed loop. If we apply the latter fact to a rectangular circuit with two sides of length $\epsilon \ll R$ with radial direction and the other two sides oriented vertically, we get that $v_r(r)$ is independent of the distance x' from the bottom of the container, when r and t are fixed (up to corrections of the order of $\frac{x}{R}$, which are negligible)(More formally, since the lenght of the horizontal sides is much smaller than the cylinder's radius, v_r can be considered to be constant on each side; neglecting vertical motion, the total circularion is $\epsilon \Delta v_r = 0$, from which the conclusion mentioned above follows). Therefore, going back to the continuity equation, we see that

$$\int_0^{x(t)} v_r(x', r)\mathrm{d}x' = -\frac{1}{2}r\dot{x} = xv_r(r) \Rightarrow v_r = -\frac{r\dot{x}}{2x}$$

Therefore, we can calculate the kinetic energy of the fluid below the cylinder:

$$K_{w1} = \frac{1}{2}\int v^2 \mathrm{d}m \approx \frac{1}{2}\int_0^R 2\pi\rho_w r x \frac{r^2\dot{x}^2}{4x^2}\mathrm{d}r = \frac{\pi\rho_w}{4x}\int_0^R r^3 \mathrm{d}r = \frac{1}{16}\pi\rho_w \frac{R^4}{x}\dot{x}^2$$

Furthermore, the kinetic energy of the remaining fluid is again negligible. This follows from the continuity equation. The water in the region above the top face of the cylinder will be approximately at rest, giving no considerable contribution to the total energy. The water moving around the cylinder will satisfy

$$\pi R^2 \dot{x} + Av = 0$$

where A is the area of the container at the level of the fluid that we're considering, minus the area of the cylinder. As long as $\frac{A}{R} \gg h$ (i.e. the walls of the container aren't incredibly close to those of the cylinder), the total kinetic energy of this portion of fluid will be proportional to

$$\frac{1}{2}AH\rho_w \left(\dot{x}\pi R^2/A\right)^2 = \frac{H\rho_w \pi R^4 \dot{x}^2}{2A}$$

which is much less than K_{w1}. Consequently, the only relevant contribution of the fluid to the total energy is $U_w + K_{w1}$.

We can now invoke conservation of energy, as mentioned at the beginning, with the additional conditions $x(t=0) = h, \dot{x}(t=0) = 0$:

$$E = \frac{1}{2}\pi R^2 H\rho_c \dot{x}^2 + \frac{1}{16}\pi\rho_w \frac{R^4}{x}\dot{x}^2 + \pi R^2 H\rho_c g x - \pi R^2 H\rho_w g x$$

$$= \pi R^2 H(\rho_c - \rho_w)gx + \frac{1}{2}\pi R^2 \dot{x}^2 \left(\rho_c H + \frac{1}{8}\rho_w \frac{R^2}{x}\right) = \pi R^2 H(\rho_c - \rho_w)gh$$

Notice that by our original assumptions, the following chain of inequalities holds:

$$\rho_c H < \rho_c R \ll \rho_w \frac{R^2}{10h} < \rho_w \frac{R^2}{8x}$$

Thus, the kinetic energy of the cylinder itself is negligible; we're left with

$$\frac{1}{16}\pi\rho_w\frac{R^4}{x}\dot{x}^2 \approx \pi R^2 gH(\rho_c-\rho_w)(h-x) \Rightarrow \dot{x} = -4\sqrt{\frac{gH}{R^2}\left(\frac{\rho_c}{\rho_w}-1\right)x(h-x)}$$

where the negative sign comes from the fact that the distance diminishes with time. We can then separate variables and integrate:

$$\int_h^0 \frac{\mathrm{d}x}{\sqrt{x(h-x)}} = -\int_0^{\pi/2} \frac{2h\sin\theta\cos\theta\mathrm{d}\theta}{\sqrt{h^2\sin^2\theta(1-\sin^2\theta)}} = -\pi$$

$$= -4\sqrt{\frac{gH}{R^2}\left(\frac{\rho_c}{\rho_w}-1\right)}\int_0^T \mathrm{d}t = -4T\sqrt{\frac{gH}{R^2}\left(\frac{\rho_c}{\rho_w}-1\right)}$$

where the substitution $x = h\sin^2\theta$ has been used. In conclusion, the time the cylinder needs to reach the bottom of the container is

$$T = \frac{\pi R}{4\sqrt{gH\left(\frac{\rho_c}{\rho_w}-1\right)}}$$