Physics Cup 2023 - Problem 1

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1 Introduction

This solution is organized as follows: first, I will qualitatively describe the way that the system evolves. I will use the given conditions to justify a set of reasonable approximations that will greatly simplify the solution. Afterwards, I will use the principle of conservation of energy to find an expression for the falling time.

2 Qualitative description

Since everything that happens is radially symmetric with respect to the axis of the cylinder, we will work in a system described by two coordinates: r, the radial distance from the axis of the cylinder, and z, the height measured from the floor of the container.

The viscosity of water is considered insignificant in this problem; for this reason, I will be able to use conservation of energy to analyze the phenomena that take place.

2.1 Beneath the cylinder

As it falls, the cylinder will set the water beneath and around it into motion. Let us first study the way that the water directly beneath the cylinder moves.

- Due to the incompressibility of water, it's clear that the liquid beneath the cylinder will have a radial velocity outwards, to ensure that the water displaced by the descending cylinder is transported outside the space between the cylinder and the floor. Let this radial velocity be $v_r(r)$ (the velocity also depends on z, but it will be proven shortly that this dependence is quantitatively insignificant), u be the downwards velocity of the cylinder, and z_c be the z-coordinate of the base of the cylinder. The aforementioned conservation of water volume requires that, for the water leaving a cylinder of radius r and height z_c beneath the solid cylinder,

$$\pi r^2 u = 2\pi r z_c v_r(r) \Longrightarrow v_r(r) = \frac{r}{2z_c}u. \tag{1}$$

  We know that $\rho_w R \gg 10\rho_c h$, while $\rho_c > \rho_w$, $h \geq z_c$, so that $R \gg 10h > 10z_c$. Meanwhile, $r \in [0, R]$, which means that for almost all[1] values of r, $v_r(r) \gg u$.

[1] This means that the set of values of r for which the above is false is quantitatively insignificant.

- What about the vertical velocity of the water? And what about the dependence of the radial velocity on z? The vertical velocity of the water will be no larger than u in absolute value - it will be $-u$ precisely at the bottom edge of the cylinder, and it will taper off to 0 at $z = 0$. The general expression of the incompressibility condition is

$$\nabla \cdot \mathbf{v} = 0 \Longrightarrow \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial z}v_z = 0. \tag{2}$$

  Integrating this from $z = 0$ to $z = z_c$, we find that

$$\frac{1}{r}\Delta(rv_r) = u, \tag{3}$$

  where $\Delta(rv_r) = rv_r(r, z_c) - rv_r(r, 0) = r(v_r(r, z_C) - v_r(r, 0))$. Hence, the difference between the maximum and minimum values of v_r at a given r is u, which, as proven above, is almost always much smaller than v_r. Hence, the variation of v_r with z is insignificant and can be ignored when it comes to calculation of the kinetic energy of the water - as can the vertical velocity of the water.

According to the above reasoning, the kinetic energy of the water beneath the cylinder is

$$\begin{aligned} K_w &= \int_0^R \rho_w(2\pi r z_c dr)\frac{v_r^2(r)}{2} \\ &= \frac{\rho_w \pi u^2}{4z_c}\int_0^R r^3 dr \\ &= \frac{\rho_w \pi R^4}{16 z_c}u^2. \end{aligned} \tag{4}$$

2.2 In the rest of the water

The water outside the narrow space between the lower end of the cylinder and the floor is also set in motion by the cylinder. We can consider the kinetic energy of the water to be caused by an added mass, and we can reasonably estimate an upper bound for this added mass: as $H < R$, in order of magnitude, the added mass is not greater than about $M_{add} \lessapprox \rho_w \pi R^2 \cdot R = \rho_w \pi R^3$.[2]

[2] This comes about for the following reason: the volume of water that is displaced has a cross-section area roughly equal to that of the cylinder, and its length cannot be much larger than the largest of H and the rough linear size of the cross-section, which in this case is a circle of radius R (if it were significantly larger than the second of the above, then a very thin disk passing through water would cause a long cylinder of water to move along with it, which is clearly false).

How does its kinetic energy compare to that of the water beneath the cylinder? The kinetic energy of this added mass is

$$K_{out} = \frac{M_{add}u^2}{2} \lessapprox \frac{\rho_w \pi R^3}{2} u^2, \tag{5}$$

so that the ratio between the kinetic energies of the two bodies of water is

$$\frac{K_{out}}{K_w} \lessapprox \frac{8z_c}{R} \ll 1. \tag{6}$$

Hence, the motion of the water outside the narrow space mentioned above is insignificant.

3 Calculation of the time

- We can express the potential energy of the system if we consider the solid cylinder to be a cylinder of density $\rho_c - \rho_w$ in a medium of constant density ρ_w. We find that it is

$$U = (\rho_c - \rho_w)\pi R^2 H g z_c \tag{7}$$

  with respect to the state where the cylinder is in contact with the floor.

- The total kinetic energy of the system is

$$\begin{aligned} K &= K_w + K_{add} + K_{cylinder} \\ &= \frac{\rho_w \pi R^4}{16 z_c} u^2 + K_{add} + \frac{\rho_c \pi R^2 H}{2} u^2. \end{aligned} \tag{8}$$

  According to what was said above, K_{add} is insignificant, and

$$\frac{K_{cylinder}}{K_w} = \frac{8\rho_c z_c}{\rho_w R} \cdot \frac{H}{R} \ll 1, \tag{9}$$

  so that the only significant part of K is K_w.

- Initially, $z_c = h$ and $u = 0$. Hence, conservation of energy compels that

$$\begin{aligned} &(\rho_c - \rho_w)\pi R^2 H g z_c + \frac{\rho_w \pi R^4}{16 z_c} u^2 = (\rho_c - \rho_w)\pi R^2 H g h \\ \Longrightarrow\ & u = \left(\frac{4}{R}\sqrt{\frac{\rho_c - \rho_w}{\rho_w} H g}\right)\sqrt{z_c(h - z_c)} \\ \Longrightarrow\ & \frac{dz_c}{\sqrt{z_c(h - z_c)}} = -\frac{4}{R}\sqrt{\frac{\rho_c - \rho_w}{\rho_w} H g}\, dt, \end{aligned} \tag{10}$$

where I have used the fact that $u = -dz_c/dt$. Integrating this from $z_c = h$ to $z_c = 0$, we find that the time taken for the cylinder to fall to the floor is

$$
\begin{aligned}
-\int_0^h \frac{dz_c}{\sqrt{z_c(h-z_c)}} &= -\frac{4}{R}\sqrt{\frac{\rho_c-\rho_w}{\rho_w}Hg}\,T \\
\Longrightarrow \pi &= \frac{4}{R}\sqrt{\frac{\rho_c-\rho_w}{\rho_w}Hg}\,T \\
\Longrightarrow T &= \frac{\pi R}{4}\sqrt{\frac{\rho_w}{\rho_c-\rho_w}\cdot\frac{1}{Hg}}.
\end{aligned}
\tag{11}
$$

Interestingly, the time does not depend on h.

Note: The integral above can be calculated as follows. Let $u = \arcsin\sqrt{1-\frac{z_c}{h}}$. Then

$$
\sin u = \sqrt{1-\frac{z_c}{h}} \Longrightarrow \cos u\, du = -\frac{1}{2h}\cdot\frac{1}{\sqrt{1-\frac{z_c}{h}}}\,dz_c \Longrightarrow dz_c = -2h\sin u\cos u\, du. \tag{12}
$$

Additionally, $\cos u = \sqrt{1-\sin^2 u} = \sqrt{\frac{z_c}{h}}$, so that, plugging the above into the integral, we get

$$
\begin{aligned}
\int_0^h \frac{dz_c}{\sqrt{z_c(h-z_c)}} &= -2\int_{z_c=0}^{z_c=h} du \\
&= -2\big(\arcsin(1-1) - \arcsin(1-0)\big) \\
&= 2\cdot\frac{\pi}{2} \\
&= \pi.
\end{aligned}
\tag{13}
$$