Physics cup 2024 problem 1

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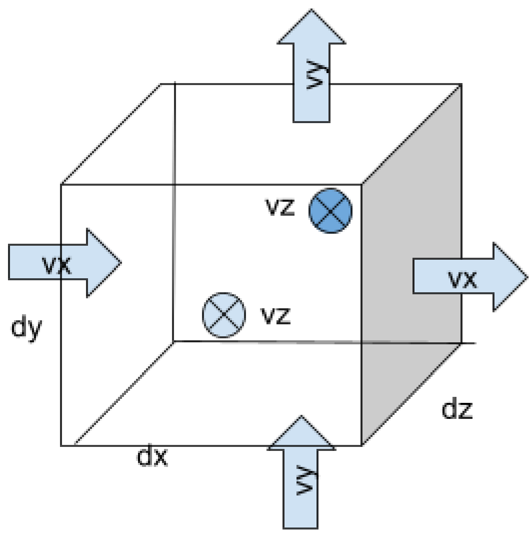
There are two methods of solving this problem, the first one is done by solving Laplace's equation, and the second one is done by making an analogy to the magnetic field around a superconductor. While I have used the first method for my initial submission, I think it is important to include the second one as it is less complex mathematically.

Method 1:

First, to describe the motion of water around the disk, I did the following:

Figuring out the motion of the water

Water, as we know, is incompressible. So for each volume element of it, the water that gets in a dx*dy*dz volume is the same as the water that gets out



$$(v_{y_{(y)}} - v_{y_{y+dy}})dxdz + (v_{x_{(x)}} - v_{x_{x+dx}})dydz + (v_{z_{(z)}} - v_{z_{z+dz}})dxdy = 0$$

$$\frac{dv_y}{dy} + \frac{dv_x}{dx} + \frac{dv_z}{dz} = 0$$

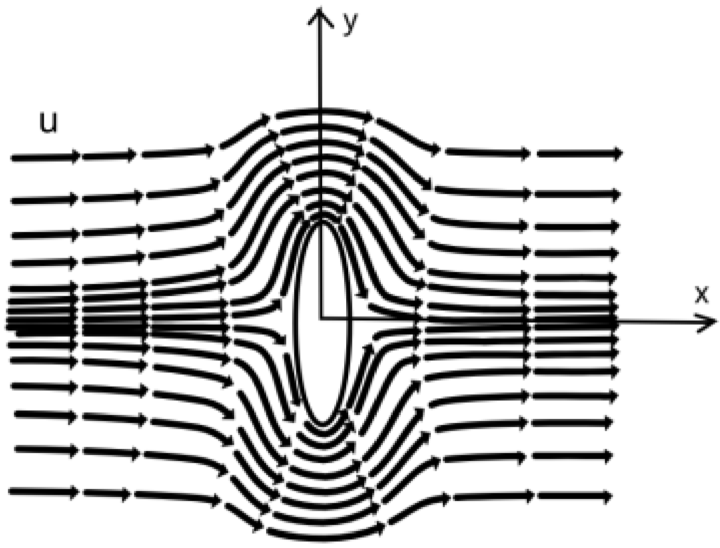
Let's define a velocity potential, ϕ, so

$$\frac{d\phi}{dy} = v_y \,, \frac{d\phi}{dx} = v_x \,, \frac{d\phi}{dz} = v_z$$

Therefore:

$$\frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dz^2} = 0$$

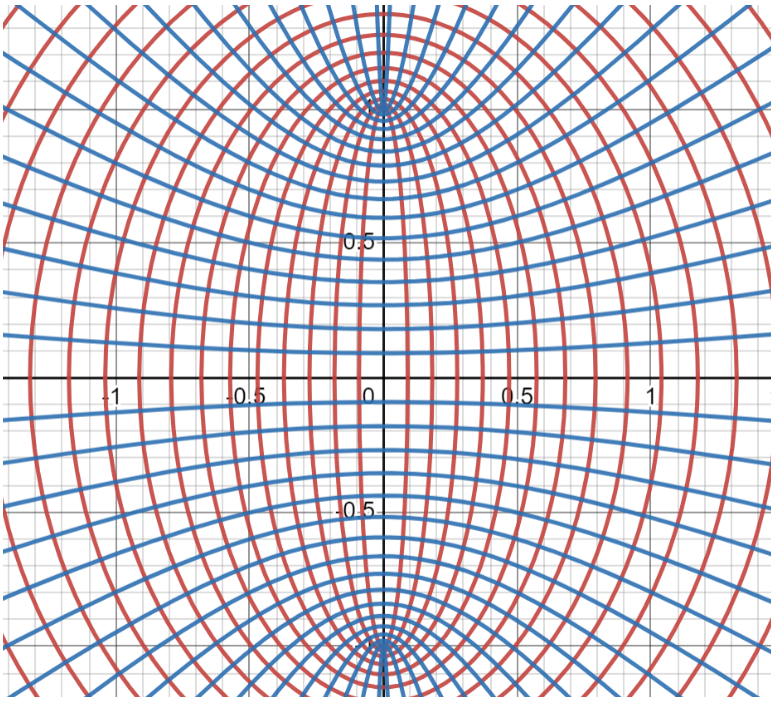
In order to find the potential flow around the disk, we will first look at the general case of flow around an ellipsoid, and make the length of its main axis approach 0.



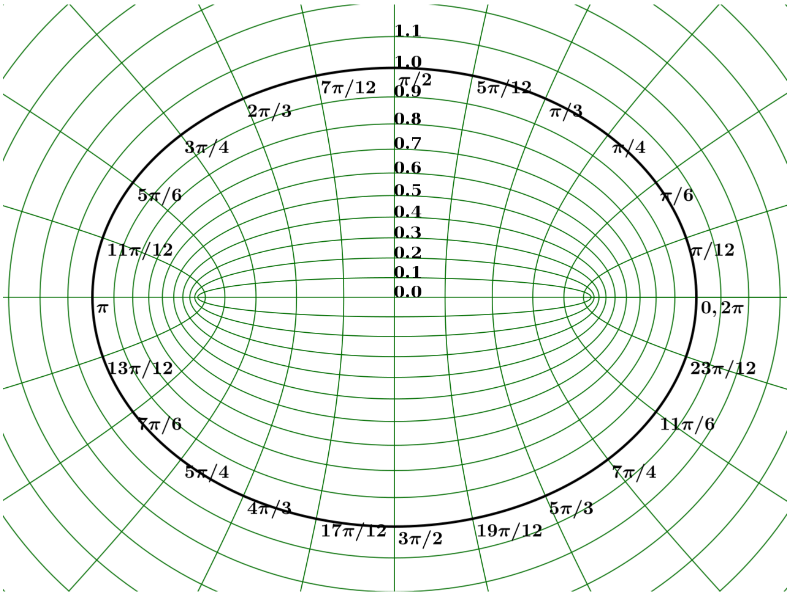
the flow around an ellipse

We would first start by establishing a 3d elliptical coordinate system, then establish boundary conditions and solve the laplace equation.

Since a spheroid is the rotation shape of an ellipse, let's make this a 2d elliptic coordinate system with the third dimension being rotation around the y axis.



a 2D elliptical coordinate system, blue lines are of constant ν (the η axis) and red lines are of constant η(the ν axis). Made with desmos



elliptical coordinate system, from Wikipedia

Elliptical coordinates are defined such as when the axis η is at a constant value, x and y create a ellipses, and when the axis ν is at a constant value, x and y create hyperbolae with the same focal points

$x_0 = k * sinh(\eta)sin(\nu)$

$y_0 = k * cosh(\eta)cos(\nu)$

where k is a constant of scale

so

$\frac{x^2}{sinh^2(\eta)} + \frac{y^2}{cosh^2(\eta)} = sin^2(\nu) + cos^2(\nu) = 1$

and

$\frac{x^2}{sin^2(\nu)} - \frac{y^2}{cos^2(\nu)} = sinh^2(\eta) - cosh^2(\eta) = 1$

Introducing an angle ω from the y to z axis, and to define the ellipsoid we would convert x_0 from the previous system to the distance from the x axis $r = x_0 = x_0cos(\varphi)\,\hat{x}_1 + x_0sin(\omega)\,\hat{z}_1$

$y_1 = k * cosh(\eta)sin(\nu)cos(\omega)$

$x_1 = k * sinh(\eta)cos(\nu)$

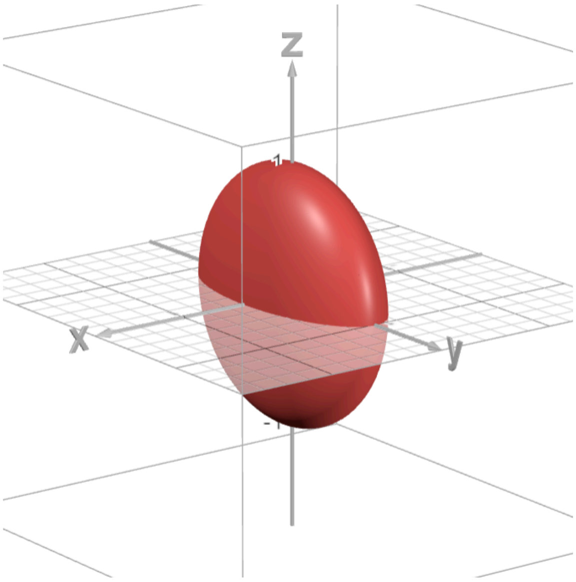
$z_1 = k * cosh(\eta)sin(\nu)sin(\omega)$

we simplify according to $sin^2(\nu) + cos^2(\nu) = 1$, and $sinh^2(\mu) - cosh^2(\mu) =- 1$

$y_1 = k\sqrt{1 + \zeta^2}\sqrt{1 - \mu^2}cos(\omega)$

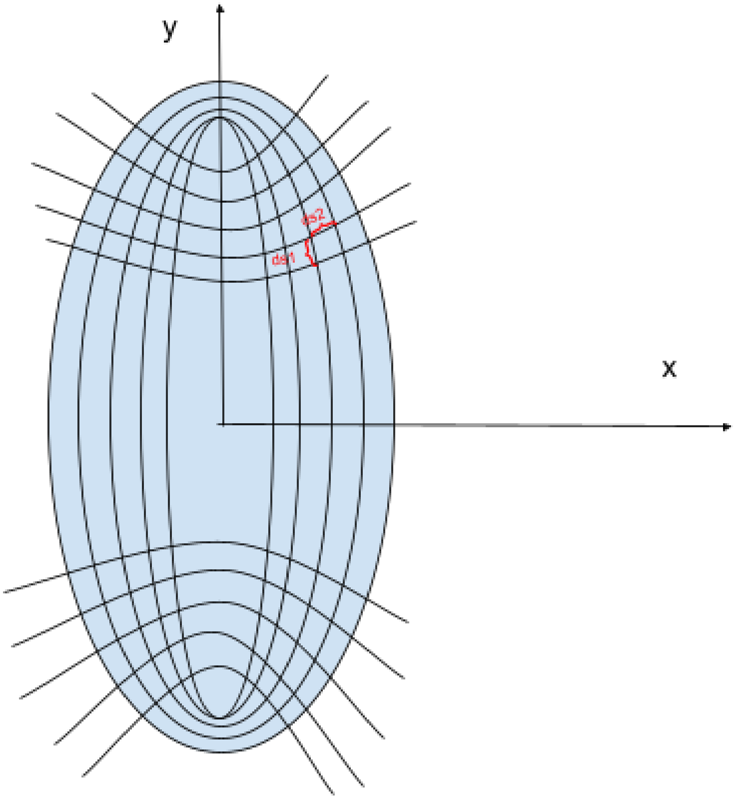
$x_1 = k\zeta\mu$

$z_1 = k\sqrt{1 + \zeta^2}\sqrt{1 - \mu^2}sin(\omega)$



in the figure, an ellipsoid defined by $\frac{x^2}{sinh^2(\eta)} + \frac{y^2+z^2}{cosh^2(\eta)} = 1$, made with desmos 3d

We now define a length factor for each of our new coordinates, the length factor is defined as the distance from two points at a curtain place in our coordinate system



$$\frac{ds_\zeta}{d\zeta} = \sqrt{(\frac{dx}{d\zeta})^2 + (\frac{dy}{d\zeta})^2 + (\frac{dz}{d\zeta})^2} = k\sqrt{(\frac{\zeta\sqrt{1-\mu^2}}{\sqrt{1+\zeta^2}}cos(\omega))^2 + (\mu)^2 + (\frac{\zeta\sqrt{1-\mu^2}}{\sqrt{1+\zeta^2}}sin(\omega))^2}$$

$$= k\sqrt{(\frac{\zeta^2(1-\mu^2)+\mu^2(1+\zeta^2)}{1+\zeta^2}} = k\sqrt{\frac{\zeta^2+\mu^2}{\zeta^2-1}}$$

$$\frac{ds_\mu}{d\mu} = \sqrt{(\frac{dx}{d\mu})^2 + (\frac{dy}{d\mu})^2 + (\frac{dz}{d\mu})^2} = k\sqrt{(\frac{-\mu\sqrt{1+\zeta^2}}{\sqrt{1-\mu^2}}cos(\omega))^2 + (\zeta)^2 + (\frac{-\mu\sqrt{1+\zeta^2}}{\sqrt{1-\mu^2}}sin(\omega))^2} =$$

$$k\sqrt{\frac{\mu^2(1+\zeta^2)+(\zeta)^2(1-\mu^2)}{1-\mu^2}} = k\sqrt{\frac{\zeta^2+\mu^2}{1-\mu^2}}$$

$$\frac{ds_\omega}{d\omega} = \sqrt{(\frac{dx}{d\omega})^2 + (\frac{dy}{d\omega})^2 + (\frac{dz}{d\omega})^2} = k\sqrt{(\sqrt{1 + \zeta^2}\sqrt{1 - \mu^2}sin(\omega))^2 + (\sqrt{1 + \zeta^2}\sqrt{1 - \mu^2}cos(\omega))^2} =$$

$$k\sqrt{(1 + \zeta^2)(1 - \mu^2)} = \sqrt{\zeta^2 + 1}\sqrt{1 - \mu^2}$$

using these elements, we will define the new velocity potential ϕ as

$$\frac{d\phi}{ds_\mu} = v_\mu \quad , \frac{d\phi}{ds_\zeta} = v_\zeta \quad , \frac{d\phi}{ds_\omega} = v_\omega$$

Now we can do the same thing we did in part 1, but each area element will be the product of two length elements,

$$\frac{d}{d\mu}(\frac{ds_\omega}{d\omega}\frac{ds_\zeta}{d\zeta}\frac{d\phi}{ds_\mu}) + \frac{d}{d\zeta}(\frac{ds_\omega}{d\omega}\frac{ds_\mu}{d\mu}\frac{d\phi}{ds_\zeta}) + \frac{d}{d\omega}(\frac{ds_\zeta}{d\zeta}\frac{ds_\mu}{d\mu}\frac{d\phi}{ds_\omega}) = 0$$

Because of symmetry, we know $v_\omega = 0$

$$\frac{d}{d\mu}(\frac{ds_\omega}{d\omega}\frac{ds_\zeta}{d\zeta}\frac{d\mu}{ds_\mu}\frac{d\phi}{d\mu}) + \frac{d}{d\zeta}(\frac{ds_\omega}{d\omega}\frac{ds_\mu}{d\mu}\frac{d\zeta}{ds_\zeta}\frac{d\phi}{d\zeta}) = 0$$

$$\frac{d}{d\mu}((1 - \mu^2)\frac{d\phi}{d\mu}) + \frac{d}{d\zeta}((\zeta^2 + 1)\frac{d\phi}{d\zeta}) = 0$$

we will solve this with separation of variables

$$\phi = ukA_{(\zeta)}B_{(\mu)}$$

$$\frac{\frac{d}{d\mu}((1-\mu^2)\frac{dB}{d\mu})}{B} = c$$

$$c = \alpha(\alpha + 1)$$

$$\frac{d}{d\mu}((\mu^2 - 1)\frac{dB}{d\mu}) = \alpha(\alpha + 1)B$$

$$B = \sum_{n=-\infty}^{\infty} a_n\mu^n$$

$$\frac{d}{d\mu}(\sum_{n=-\infty}^{\infty} na_n\mu^{n+1} - \sum_{n=-\infty}^{\infty} na_n\mu^{n-1}) = \alpha(\alpha + 1)\sum_{n=-\infty}^{\infty} a_n\mu^n$$

$$\sum_{n=-\infty}^{\infty}(n + 1)na_n\mu^n - \sum_{n=-\infty}^{\infty} n(n - 1)a_n\mu^{n-2} = \alpha(\alpha + 1)\sum_{n=-\infty}^{\infty} a_n\mu^n$$

$$a_0 = a_0$$

$$a_0\alpha(\alpha - 1) =- 2a_2$$

$$a_1\alpha(\alpha - 1) = 2a_1 - 6a_3$$

$$a_{n-2}(- \alpha(\alpha - 1) + (n - 2)(n - 1)) = a_n(n)(n - 1)$$

$$a_{n-2} = \frac{a_n(n)(n-1)}{(n-2)(n-1)-\alpha(\alpha+1)}$$

$$a_{n+2} = \frac{a_n(n(n+1)-\alpha(\alpha+1))}{(n+2)(n+1)}$$

The solution to this type of equation is a well known one, this is Legendre's differential equation, and it can be solved with the Legendre functions of the first and second kind

$$B = (a \cdot P_\alpha(\mu) + b \cdot Q_\alpha(\mu))$$

These unique functions are used everywhere from quantum mechanics to Mie scattering.

We know that when $\zeta \to \infty$, the velocity towards the μ direction in an almost straight angle ($\mu \to 1$) the velocity of the water is u to the x direction, therefore it is u times $\frac{\sqrt{y^2+z^2}}{\sqrt{x^2+y^2+z^2}} \approx_{\zeta\to\infty} \sqrt{1-\mu^2}$, to the μ direction. we also know because of that when $\zeta \to \infty$, the velocity doesn't jump up to infinity, therefore the velocity potential function is only limited to the first order, so $\alpha = 1$

$$a_{n+2} = \frac{a_n(n(n+1)-2)}{(n+2)(n+1)} = \frac{a_n(n+2)(n-1)}{(n+2)(n+1)} = \frac{a_n(n-1)}{(n+1)}$$

$$P_1(x) = a_0 x$$

$$a_{-n-2} = \frac{a_n(-n-1)}{(-n-3)} = \frac{a_{n+2}(-n+1)}{(-n-3)} = \ldots = \frac{a_0(0+1)}{(-n-3)} \ or \ \frac{a_1(-1+1)}{(-n-3)}$$

for n<0

$$a_{-2n} = \frac{a_0}{(-2n-1)}$$

$$a_{-2n-1} = 0$$

for n=0:

$$a_{n=0} = \frac{a_0}{(0-1)} = -a_{n=0}$$

$$a_{n=0} = 0$$

$$a_{-n-2} = \frac{a_{-n}(-n-1)}{(-n-3)} = \frac{a_{-n+2}(-n+1)}{(-n-3)} = \ldots = \frac{-a_{-2}}{(-n-3)} \ or \ \frac{a_{-1}(-1+1)}{(-n-3)}$$

$$Q_1(x) = -\sum_{n=1}^{\infty} \frac{a_{-2}}{x^{2n}(2n+1)}$$

Let's write our boundary condition for μ

$$\frac{d\phi}{ds_\mu}_{\mu\to1,\zeta\to\infty} = \frac{\frac{d\phi}{ds_\mu}_{\mu=1,\zeta\to\infty}}{k\sqrt{\frac{\zeta^2+\mu^2}{1-\mu^2}}} = u\sin(\nu) = u\sqrt{1-\mu^2}$$

$$(a + b\sum_{n=1}^{\infty} \frac{-2n*\mu^{-2n-1}}{(2n+1)})\sqrt{1-\mu^2} = const * \sqrt{1-\mu^2}$$

$$(a - \infty b) = const$$

$$b = 0$$

$$B = a\mu$$

For the function A, the solution is similar, we just need to add the imaginary unit to ζ

$$A = (c \cdot P_1(i\zeta) + d \cdot Q_1(i\zeta)) =$$

$$(c \cdot i\zeta + d \cdot (-\sum_{n=1}^{\infty} \frac{a_{-2}}{(i\zeta)^{2n}(2n+1)})) = (c \cdot i\zeta + d \cdot (a_{-2} - \sum_{n=0}^{\infty} \frac{a_{-2}}{(i\zeta)^{2n}(2n+1)})) =$$

$$(c \cdot i\zeta - d \cdot (\zeta i(\sum_{n=0}^{\infty} \frac{a_{-2}}{(i\zeta)^{2n+1}(2n+1)}) - a_{-2}))$$

$$(c \cdot i\zeta - d \cdot (i\zeta((\frac{a_{-2}}{(i\zeta)\ (1)}) + (\frac{a_{-2}}{(-i\zeta^3)\ (3)}) + (\frac{a_{-2}}{(i\zeta^5)\ (5)}) + (\frac{a_{-2}}{(-i\zeta^7)\ (7)})\ldots) - a_{-2})) =$$

$$(c \cdot i\zeta - d \cdot a_{-2}(\zeta((\frac{1}{(\zeta)\ (1)}) - (\frac{1}{(\zeta^3)\ (3)}) + (\frac{1}{(\zeta^5)\ (5)}) - (\frac{1}{(\zeta^7)\ (7)})\ldots) - 1)) =$$

$$(c \cdot i\zeta - d \cdot a_{-2}(\zeta\int((\frac{1}{(\zeta^2)}) - (\frac{1}{(\zeta^4)}) + (\frac{1}{(\zeta^6)}) - (\frac{1}{(\zeta^8)})\ldots)d(\frac{1}{\zeta}) - 1)) =$$

$(c \cdot i\zeta - d \cdot a_{-2}(\zeta\int(\frac{\frac{1}{\zeta^2}}{1+\frac{1}{\zeta^2}})d(\frac{1}{\zeta}) - 1)) =$

$(c \cdot i\zeta - d \cdot a_{-2} \cdot (\zeta tan^{-1}(\frac{1}{\zeta}) - 1)) = (c \cdot i\zeta + d \cdot (1 - \zeta cot^{-1}(\zeta)))$

To find c and d, we know that when $\zeta \to \infty$, the velocity towards the ζ direction is u times.

$\frac{x}{\sqrt{x^2+y^2+z^2}} \approx_{\zeta\to\infty} \mu,$

$\frac{d\phi}{ds_\zeta}_{\mu\to1,\zeta\to\infty} = u\mu$

$uk\frac{d\phi}{d\zeta}\frac{1}{k}\sqrt{\frac{\zeta^2-1}{\zeta^2+\mu^2}}_{\mu\to1,\zeta\to\infty} = u\mu$

$ua\mu \cdot \sqrt{\frac{\zeta^2-1}{\zeta^2+1}} \cdot (c \cdot i\zeta + d \cdot (1 - \zeta cot^{-1}(\zeta)) = u\mu$

$a \cdot (c \cdot i + d \cdot (1 - 1)) = 1$

$c = \frac{1}{ai}$

$A = d \cdot (1 - \zeta \cdot cot^{-1}(\zeta)) + \frac{1}{ai} \cdot i\zeta$

$\phi = uk\mu a(d \cdot (1 - \zeta \cdot cot^{-1}(\zeta)) + \frac{1}{a}\zeta) = uk\mu(g_1 \cdot (\zeta \cdot cot^{-1}(\zeta) - 1) + \zeta),\ g_1 =- da$

*** note: the choice of variables is not meant to endorse or make fun of the United Kingdom, any mention of the letters uk is coincidental only.

Now let's find the parameter g_1:

We know that the points where $\zeta = \zeta_0$, form an ellipsoid in our 3D space. To describe the flow around it we can impose a boundary condition that no water at the surface of the ellipsoid moves perpendicular to it.

$\frac{d\phi}{ds_\zeta}_{\zeta=\zeta_0} = 0$

$\frac{d\phi}{d\zeta}\frac{1}{k}\sqrt{\frac{\zeta^2-1}{\zeta^2+\mu^2}}_{\zeta=\zeta_0} = 0$

$\frac{d\phi}{d\zeta}_{\zeta=\zeta_0}\frac{1}{k}\sqrt{\frac{\zeta_0^2-1}{\zeta_0^2+\mu^2}} = 0$

$uk\mu(g_1 \cdot (cot^{-1}(\zeta) - \frac{\zeta}{\zeta^2+1}) + 1)_{\zeta=\zeta_0}\frac{1}{k}\sqrt{\frac{\zeta_0^2-1}{\zeta_0^2+\mu^2}} = 0$

$(g_1 \cdot (cot^{-1}(\zeta_0) - \frac{\zeta_0}{\zeta_0^2+1}) + 1)\sqrt{\frac{\zeta_0^2-1}{\zeta_0^2+\mu^2}} = 0$

$g_1 \cdot (cot^{-1}(\zeta_0) - \frac{\zeta_0}{\zeta_0^2+1}) =- 1$

$$g_1 = \frac{1}{\left(\frac{\zeta_0}{\zeta_0^2+1} - cot^{-1}(\zeta_0)\right)}$$

If we choose now to make $\zeta_0 \rightarrow 0$, our new ellipsoid will now become:

$$\frac{x^2}{sinh^2(\eta)} + \frac{y^2+z^2}{cosh^2(\eta)} = k^2$$

$$\frac{x^2}{\zeta_0^{\ 2}} + \frac{y^2+z^2}{\zeta_0^{\ 2}+1} = k^2$$

$$x_{max,min} = \pm\ k\zeta_0 \ \rightarrow 0$$

$$y^2 + z^2 < k^2$$

we got a disk with radius k.

Now let's push g_1 to the limit where $\zeta_0 \rightarrow 0$

$$g_1 = \frac{1}{\left(\frac{0}{0^2+1} - cot^{-1}(0)\right)} = \frac{-2}{\pi}$$

$$\phi = uk\mu\left(\frac{2}{\pi}\left(1 - \zeta \cdot cot^{-1}(\zeta)\right) + \zeta\right)$$

In the lab's frame of reference, the water far away is stationary and only the water near the disk moves. To move to that reference frame we need to subtract a uniform flow in the x direction from our potential function:

$$\phi_{uniform} = xu = uk\zeta\mu$$

$$\phi = uk\mu\left(\frac{2}{\pi}\left(1 - \zeta \cdot cot^{-1}(\zeta)\right)\right)$$

Now, to find the force enacted on the plate to the x direction, we will integrate the pressure over the surface of the disk, the pressure is defined as

$$P = \rho\left(\frac{d\phi}{dt} + \frac{1}{2}\nabla\phi^2\right)$$

at the moment of impact with the water, the change in velocity (and thus in potential) is almost instantaneous, therefore we can neglect any other term in the

$$F = -\ \rho\int_0^{2\pi}\int \left.\frac{d\phi}{dt}\right|_{\zeta_0} ds_\mu ds_\omega =$$

$$-\ \rho\dot{u}k^3\int\int \mu\left(\frac{2}{\pi}\left(1 - \zeta_0 \cdot cot^{-1}(\zeta_0)\right)\right)\sqrt{\frac{\zeta_0^{\ 2}+\mu^2}{1-\mu^2}}\sqrt{\zeta_0^{\ 2} + 1}\sqrt{1 - \mu^2}\,d\mu d\omega =$$

$$-\ \rho\dot{u}k^3\int_0^{2\pi}\int \mu\left(\frac{2}{\pi}\left(1 - \zeta_0 \cdot cot^{-1}(\zeta_0)\right)\right)\sqrt{\zeta_0^{\ 2} + \mu^2}\sqrt{\zeta_0^{\ 2} + 1}\,d\mu d\omega =$$

$$-\ \rho\dot{u}k^3\int_0^{2\pi}\int \mu\left(\frac{2}{\pi}(1)\right)\sqrt{0^2 + \mu^2}\sqrt{0^2 + 1}\,d\mu d\omega =$$

$$-\ \rho\dot{u}k^3\int_0^{2\pi}\int \frac{2}{\pi}\mu^2\,d\mu d\omega =$$

$$-\ 2\pi \cdot \frac{2}{\pi} \cdot \rho\dot{u}k^3\int \mu^2 d\mu =$$

Again, μ represents the cosine of the angle between the x axis and the yz plane when $\zeta \to \infty$. Since we only care about one side of the disk, the limits of integration for the angle are 0 and $\frac{\pi}{2}$, or 0 and 1 for μ.

$$-\ 4\ \cdot\ \rho\dot{u}k^3\int_0^1 \frac{\mu^3}{3} d\mu\ =$$

$$-\ \rho\frac{4}{3}\dot{u}k^3$$

this can be seen as a mass of $M_{added} = \frac{4}{3}\rho k^3$ added to the cylinder in an inelastic collision, we can write the conservation of momentum equation

$$(M_{cylinder} + M_{added})\ \cdot u_{after} = M_{cylinder} \cdot u_{before}$$

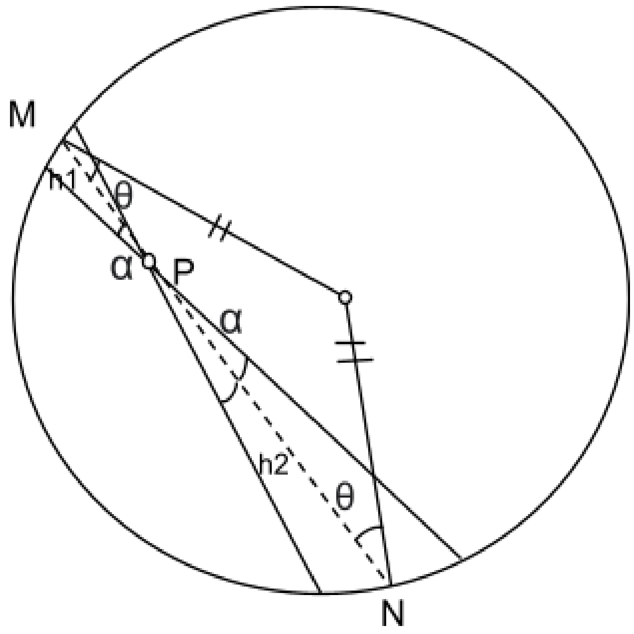
$$M_{cylinder} =\ \pi\rho_{cylinder}\ \cdot\ (k)^2\ \cdot\ H\ = \frac{\frac{4}{3}\rho k^3 u_{after}}{u_{before} - u_{after}}$$

$$\rho_{cylinder} = \frac{4\rho k u_{after}}{3\pi(u_{before} - u_{after})\cdot(H)}$$

Method 2:

the Laplace equation, or $\frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dz^2} =\ 0$ is also present in electromagnetism. In the case of electromagnetism, the velocity field is equivalent to the magnetic field around a polarized object.

We can find the added mass by solving an auxiliary problem of figuring the field around an infinitely thin circular plate. Our boundary conditions are that no water gets inside the body. In other words, the field orthogonal to the surface at the surface is zero and no movement of water happens inside the body.
This is also the behavior of the magnetic field around a superconducting object. In such an object, the surface currents arrange in such a way that the magnetic field inside is zero, therefore, if the applied field is uniform, the magnetic field created by the object inside of it will also be uniform.
We would treat the plate as an infinitely thin ellipsoid. The reason for that is that when the magnetization of it is uniform, the magnetic field caused by said magnetization is also uniform.
to prove this, we can use the electri-magnetic analogy,
we start with a homogeneously charged spherical shell, defined by the superposition of $x^2 + y^2 + z^2 = R^2$ and $x^2 + y^2 + z^2 = (R\ +\ dr)^2$
we pick a point P inside of it, and draw two narrow cones meeting at the point.

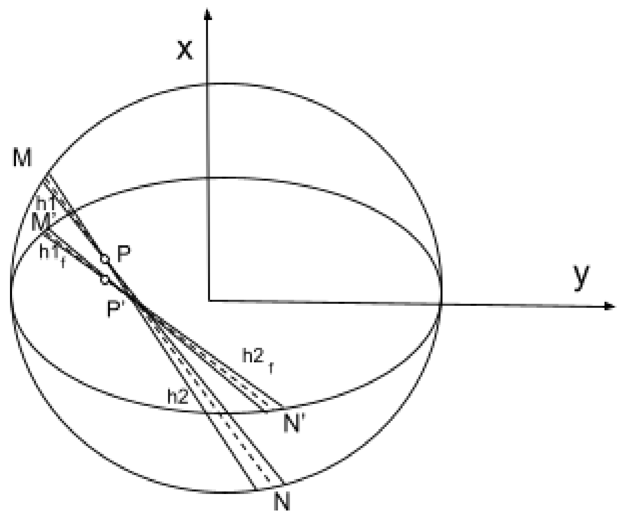


let’s find the magnitude of the electric field at the point from both surfaces:

$$E = \frac{k\rho dV_1}{h_1^2} - \frac{k\rho dV_2}{h_2^2} = \frac{k\sigma A_1}{h_1^2} - \frac{k\sigma A_2}{h_2^2} = \frac{k\sigma A_1}{h_1^2} - \frac{k\sigma A_2}{h_2^2} = \frac{k\sigma\frac{(\pi h_1^2\alpha)}{cos^2(\theta)}}{h_1^2} - \frac{k\sigma\frac{(\pi h_2^2\alpha)}{cos^2(\theta)}}{h_2^2} = 0$$

we can sum these cones over the entire surface of the sphere and get that at every point $E = 0$

Now, we pick a point P’ inside a homogeneously charged ellipsoidal shell, the shell is defined by the superposition of $\frac{x^2}{b^2} + y^2 + z^2 = R^2$ and $\frac{x^2}{b^2} + y^2 + z^2 = (R + dR)^2$. we can solve this by applying an affine transformation from the spherical case of $x \rightarrow \frac{x}{b}$



$$dV_1 = dx_1 dy_1 dz_1 \rightarrow dz_1 dy_1 \frac{dx_1}{b}$$

$$dV_2 = dx_2 dy_2 dz_2 \rightarrow dz_2 dy_2 \frac{dx_2}{b}$$

We also notice that the translation of the two bases of the cone in relation to each other, as well as the translation of the point P are parallel to the x axis,
moving the two centers to align, and we get two intersecting lines and two parallel lines creating two triangles, which we know from the Talos principle:

$$\frac{h_1}{h_2} = \frac{h_{1_f}}{h_{2_f}}$$

therefore:

$$E = \frac{k\rho dV_{1_f}}{h_{1_f}^{\ 2}} - \frac{k\rho dV_{2_f}}{h_{2_f}^{\ 2}} = \frac{k\rho dV_1}{bh_{1_f}^{\ 2}} - \frac{k\rho dV_2}{bh_{2_f}^{\ 2}} = \frac{1}{h_{1_f}^{\ 2}}\left(\frac{k\rho dV_1}{b} - \frac{k\rho dV_2 h_1^{\ 2}}{bh_2^{\ 2}}\right) = \frac{h_1^{\ 2}}{bh_{1_f}^{\ 2}}\left(\frac{k\rho dV_1}{h_1^{\ 2}} - \frac{k\rho dV_2}{h_2^{\ 2}}\right) = 0$$

we now define a new size ζ, each point has an ellipsoid of $\frac{x^2}{b^2} + y^2 + z^2 = \zeta^2$ touching it. By applying gauss's law for each layer of a uniformly charged ellipsoid, we find that the electric field inside of it is $E = c * \zeta$ normal to the surface of the ellipsoid $\frac{x^2}{b^2} + y^2 + z^2 = \zeta^2$ where c is a constant.
Therefore:

$$E_x = c * \zeta\frac{d\zeta}{dx} = c * \sqrt{\frac{x^2}{b^2} + y^2 + z^2}\frac{x}{b^2\sqrt{\frac{x^2}{b^2}+y^2+z^2}} = \frac{cx}{b^2}$$

$$E_y = c * \zeta\frac{d\zeta}{dy} = c * \sqrt{\frac{x^2}{b^2} + y^2 + z^2}\frac{y}{\sqrt{\frac{x^2}{b^2}+y^2+z^2}} = cy$$

$$E_z = c * \zeta\frac{d\zeta}{dz} = c * \sqrt{\frac{x^2}{b^2} + y^2 + z^2}\frac{z}{\sqrt{\frac{x^2}{b^2}+y^2+z^2}} = cz$$

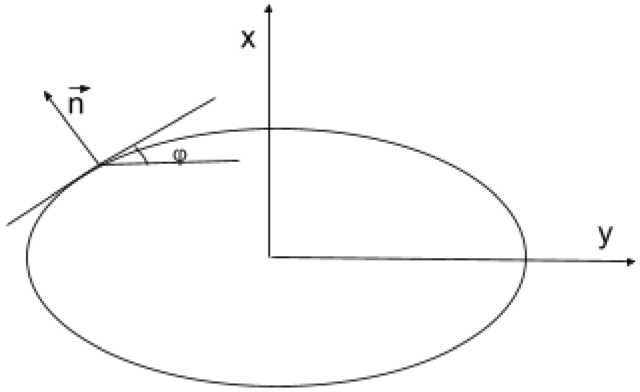
A uniformly polarized ellipsoid in the x direction is a superposition of a uniformly positively charged ellipsoid at $x = 0$ and an oppositely charged ellipsoid at $x = dx$, so for each point of (x_0, y_0, z_0) inside the ellipsoid:

$$E_x = \frac{c((x_0-0)-(x_0-dx))}{b^2} = \frac{c*dx}{b^2}, E_y = c(y_0 - y_0) = 0, E_z = c(z_0 - z_0) = 0$$

Therefore a uniformly polarized ellipsoid creates a uniform electric field inside of it.
To apply the electric-magnetic analogy, we need to consider only fields outside the body.
We can prove the magnetic field is homogeneous at an arbitrary point A inside the ellipsoid by drilling a narrow hole parallel to the magnetization axis next to it, for the points inside the hole the analogy holds and the magnetic field is uniform.
For the point A outside the hole, we know that the H field parallel to the surface is conserved, the H field is defined by the magnetic field plus $\mu_0 M$ (which is constant), therefore Since the magnetic field inside the hole is uniform, then H is uniform and the magnetic field at point A is also uniform. since this can be applied to any point in the ellipsoid, the magnetic field inside the entire ellipsoid is homogenous

We now need to find the polarization of the ellipsoid, we can do that by comparing the field produced by the polarized ellipsoid and the field exerted on it. It is known that the bound surface current is $K_m = M \times \hat{n} = M(\hat{x} \times \hat{n})$

imagine a cross section of the ellipsoid with the plane $z = 0$, from symmetry the same will apply to any plane parallel to the x axis and going through the center.



$$\frac{x^2}{b^2} + y^2 = R^2$$

$$x = \pm\, b\sqrt{R^2 - y^2}$$

the angle between the tangent of the ellipse and the y axis is φ

$$tan(\varphi) = \frac{dx}{dy} = \frac{\mp by}{\sqrt{R^2-y^2}} \Rightarrow sin(\varphi) = \frac{tan(\varphi)}{\sqrt{1+tan^2(\varphi)}} = \frac{\mp by}{\sqrt{b^2y^2+R^2-y^2}}$$

due to rotational symmetry, the $\vec{n}$ vector at the points that lay on the perimeter of the ellipse has no $\hat{z}$ component. Therefore:

$$|(\hat{x} \times \hat{n})| = sin(\varphi) = \frac{\mp by}{\sqrt{b^2y^2+R^2-y^2}}$$

$$K_m = \frac{\mp byM}{\sqrt{b^2y^2+R^2-y^2}}\hat{z}$$

Generalized to any plane parallel to the x axis and going through the center:

$$x = \pm\, b\sqrt{R^2 - r^2}, \vec{r} = \vec{y} + \vec{z}$$

$$tan(\varphi) = \frac{\mp br}{\sqrt{R^2-r^2}}$$

$$K_m = \frac{\mp brM}{\sqrt{b^2r^2+R^2-r^2}}(\hat{x} \times \hat{r})$$

To find the magnetic field, we will calculate the magnetic field at the center of the ellipsoid. Notice that the induced current on the surface is composed of current loops going around the center of the ellipsoid.

We now sum the magnetic fields generated by the loops, using the formula for the magnetic field generated by a ring at a point on its axis. Notice that the field applied from the top part is the same as the field in the bottom

$$B = \frac{\mu_0 Ir^2}{2(r^2+x^2)^{3/2}} = 2\int_0^R \frac{\mu_0 r^2 K_m \cdot \frac{dr}{cos(\varphi)}}{2(r^2+x^2)^{3/2}} = 2\int_0^R \frac{\mu_0 r^2 K_m \cdot \frac{dr}{cos(\varphi)}}{2(r^2+x^2)^{3/2}} = 2\int_0^R \frac{\mu_0 r^2 Mtan(\varphi)\cdot dr}{2(r^2+x^2)^{3/2}} = 2\int_0^R \frac{\mu_0 r^2 M\frac{br}{\sqrt{R^2-r^2}}\cdot dr}{2(r^2+b^2R^2-b^2r^2)^{3/2}}$$

in the case of a flat disc $b \to 0$

$$B = 2b\int_0^R \frac{\mu_0 r^3 M\cdot dr}{2r^3\sqrt{R^2-r^2}} = b\mu_0 M\int_0^R \frac{d(\frac{r}{R})}{\sqrt{1-(\frac{r}{R})^2}} = \frac{\pi}{2}b\mu_0 M = \frac{\pi}{2}b\mu_0\frac{m}{b*\frac{4}{3}\pi R^3} = \frac{3}{8}\mu_0\frac{m}{R^3}$$

$$m = \frac{8BR^3}{3\mu_0}$$

To find the added mass, we need to find the energy of the moving water from the reference frame of the lab. Moving to the lab's frame of reference is the same as adding to the object's frame a field of constant velocity $-\ v$. Which is akin to subtracting the applied magnetic field and only summing the potential energy from the induced magnetic field.

We know that $U = \frac{1}{2}mB = \frac{4B^2R^3}{3\mu_0}$ which is equivalent to $E = \frac{1}{2}mv^2$ since B is equivalent to v, than $\frac{2E}{B^2} = \frac{8R^3}{3\mu_0}$ is equivalent to m.

In a vacuum, the potential energy of a magnetic field per unit of volume is $\frac{dU}{dV} = \frac{B^2}{2\mu_0}$ which is equivalent to $\frac{dE}{dV} = \frac{1}{2}\rho v^2$, therefore ρ is equivalent to $\frac{1}{\mu_0}$, so the added mass is $\frac{\frac{8R^3}{3\mu_0}\rho}{\frac{1}{\mu_0}} = \frac{8R^3}{3}\rho$

Since only half of the object is immersed in water:

$m_{added} = \frac{4R^3}{3}\rho$

Just like in method 1, this can be seen as a mass of $M_{added} = \frac{4}{3}\rho R^3$ added to the cylinder in an inelastic collision, we can write the conservation of momentum equation

$(M_{cylinder} + M_{added}) \cdot u_{after} = M_{cylinder} \cdot u_{before}$

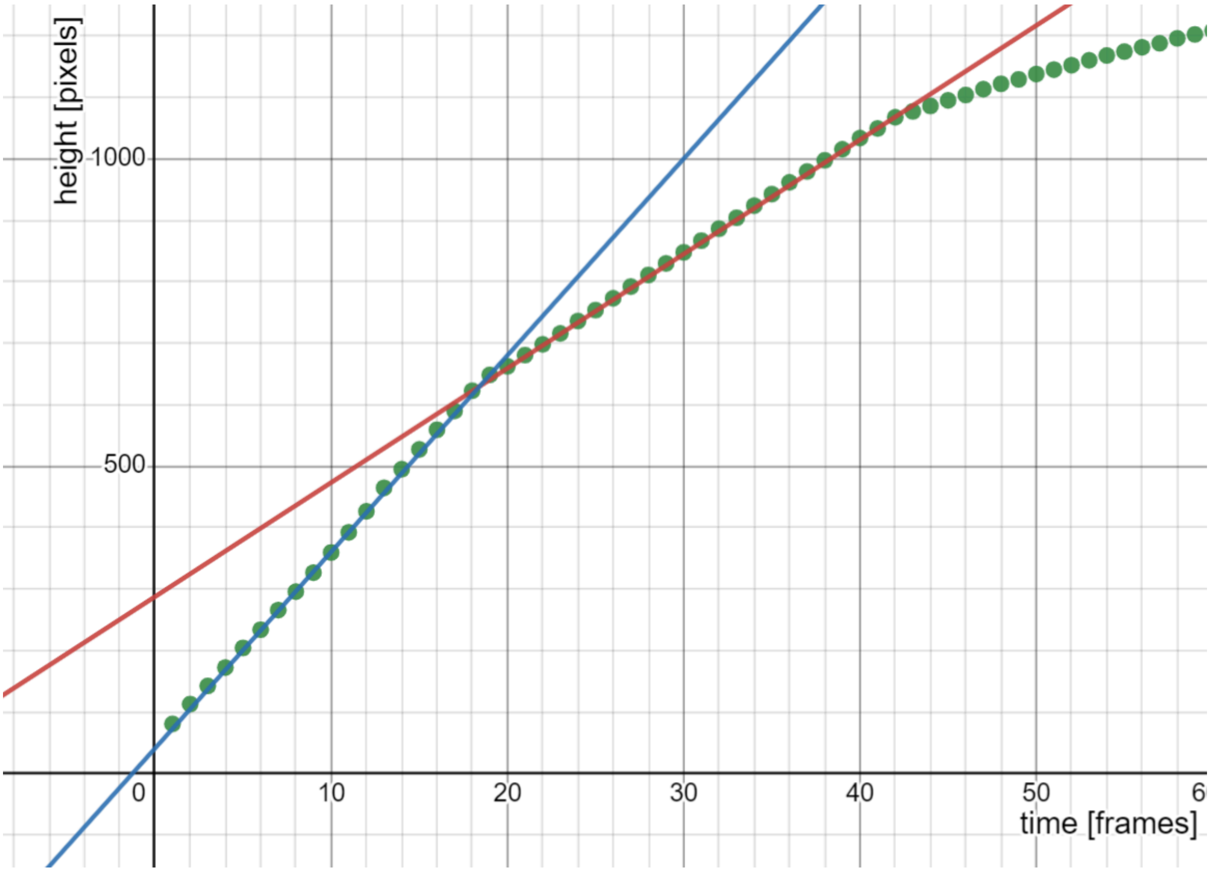
$M_{cylinder} = \pi\rho_{cylinder} \cdot (R)^2 \cdot H = \frac{\frac{4}{3}\rho R^3 u_{after}}{u_{before} - u_{after}}$

$\rho_{cylinder} = \frac{4\rho R u_{after}}{3\pi(u_{before} - u_{after})\cdot(H)}$

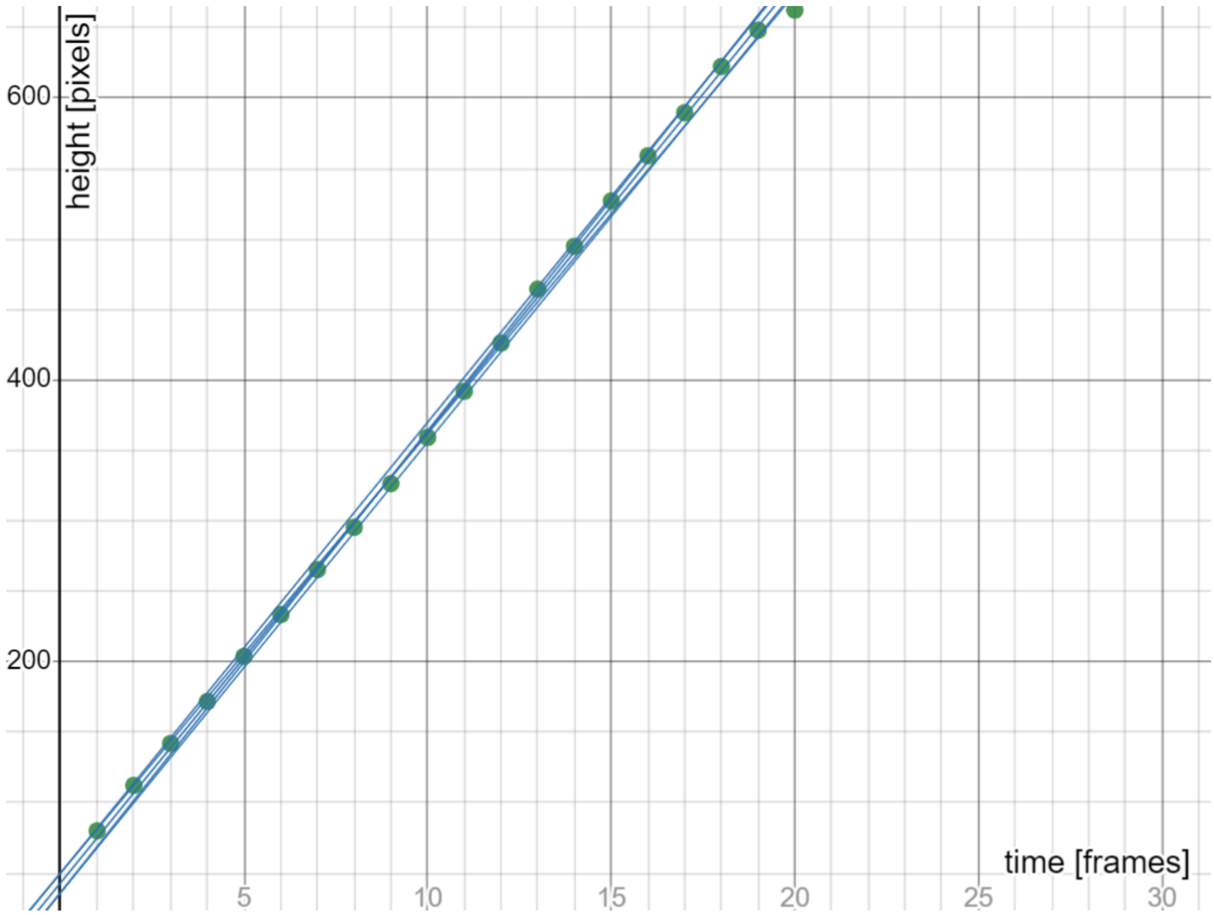
Taking measurements

I used a pixel ruler and vlc media player to measure the distance of the cylinder from the top of the screen, I took 5 measurements each frame and plotted the average in Desmos.
The disk hits the water in measurement 20 so we need to make two trend lines for before and after the impact.

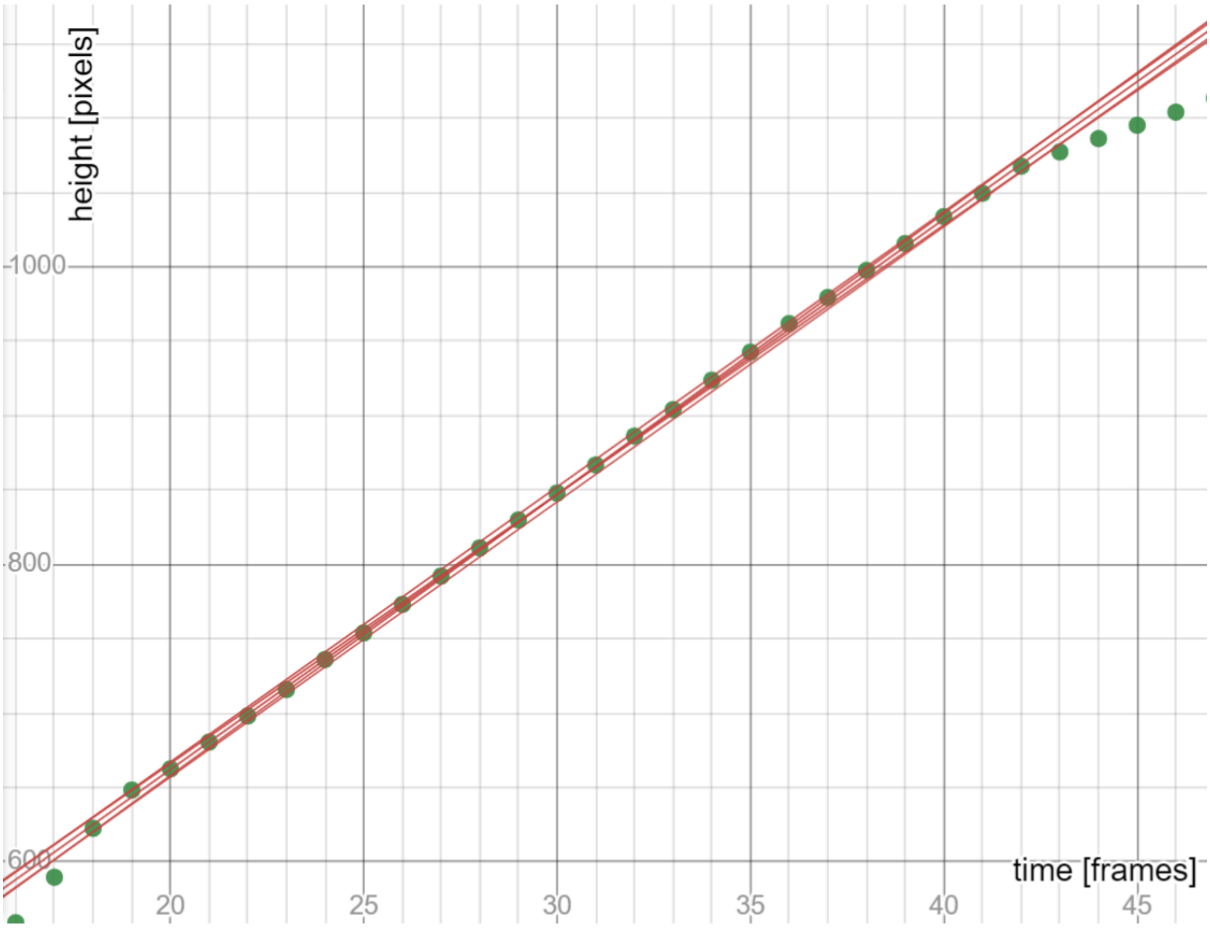
measure 1 [px]	measure 2 [px]	measure 3 [px]	measure 4 [px]	measure 5 [px]	avarage [px]	n [frame]
80	80	80	80	80	80	1
109	110	113	115	114	112.2	2
144	137	144	142	143	142	3
171	167	174	176	171	171.8	4
203	200	204	209	203	203.8	5
235	231	237	230	234	233.4	6
267	262	267	266	265	265.4	7
298	295	299	296	289	295.4	8
330	329	331	323	319	326.4	9
361	362	364	354	355	359.2	10
394	394	397	390	385	392	11
427	430	430	424	420	426.2	12
460	467	464	462	469	464.4	13
492	496	495	495	496	494.8	14
523	527	529	531	525	527	15
557	566	560	555	557	559	16
589	598	593	577	591	589.6	17
627	632	627	601	625	622.4	18
649	653	651	631	657	648.2	19
666	665	666	662	653	662.4	20
683	679	680	683	677	680.4	21
700	696	698	699	697	698	22
716	711	715	718	719	715.8	23
736	729	735	736	744	736	24
755	745	754	754	761	753.8	25
774	763	773	773	782	773	26
791	783	791	791	804	792	27
811	802	810	810	822	811	28
830	821	834	825	839	829.8	29
848	838	850	844	859	847.8	30
868	858	868	862	878	866.8	31
887	878	890	880	896	886.2	32
907	896	906	900	911	904	33
926	915	926	922	930	923.8	34
945	934	947	940	947	942.6	35
963	952	965	960	969	961.8	36
981	972	983	978	983	979.4	37
1000	989	1001	999	999	997.6	38
1017	1008	1018	1017	1018	1015.6	39
1035	1031	1034	1033	1036	1033.8	40
1051	1048	1049	1049	1050	1049.4	41
1069	1066	1068	1068	1067	1067.6	42
1079	1076	1078	1078	1075	1077.2	43
1087	1083	1091	1090	1080	1086.2	44
1094	1092	1099	1099	1092	1095.2	45
1103	1100	1107	1110	1100	1104	46
1112	1108	1116	1118	1113	1113.4	47
1123	1118	1125	1125	1120	1122.2	48
1130	1126	1132	1129	1130	1129.4	49
1137	1134	1142	1142	1135	1138	50
1143	1142	1148	1149	1144	1145.2	51
1150	1148	1156	1156	1152	1152.4	52
1160	1157	1164	1162	1159	1160.4	53



Since we only care about the first few frames after the disk hits the water (after a while instabilities develop and the system becomes hard to analyze), it is okay that the second line only matches the first few points



$u_{before} = 32.0 \pm 0.5 [px/frame]$



$u_{after} = 18.5 \pm 0.4[px/frame]$

I measured:

$k = 331[px]\ (radius\ of\ the\ disk)$

$H = 265[px]\ \ (height\ of\ the\ disk)$

$\rho_{water} = 1000[kg/m^3]$

and finally:

$\rho_{cylinder} = \frac{4\cdot(1000)\cdot331\cdot18.5}{3\cdot\pi\cdot(32.0-18.5)\cdot(265)} = 73\cdot10^1[kg/m^3]$

$\rho_{cylinder} = 73\cdot10^1 \pm 5\cdot10^1[kg/m^3]$

This result is surprising since we don't see the disk float back up at the end of the video, but considering the fact we see very little change in the velocity when the disk is underwater, it is possible that eventually it rose back to the top (even the sudden change at n=43 is duo to the bottom of the disk hitting the edge of the screen forcing me to measure the height from the top, and most likely optic effects caused the top part to appear slower, or perhaps it was an actual rotation around the x axis)