Physics cup 2024 Problem 1

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1 Introduction

The solution is organized as follows: I first describe the motion qualitatively, then I analyse the initial impact of the cylinder with the water, and from it I relate the velocities of the cylinder directly before and after impact in dependence on the cylinders density. Afterwards I use the video to make measurements using pixel and frame counting, and these measurements, together with the theoretical analysis, reveal the density of the cylinder.

2 Qualitative motion

The cylinder can first be seen falling under the influence of gravity through air before impacting the water (phase 0, $t < 0$). In the moment $t = 0$ the cylinder first impacts the water and a wave appears in the water travelling at the speed of sound in water c giving the water some momentum. The pressure gradient is high in this phase and therefore in this short amount of time τ the cylinder loses a finite amount of momentum (phase 1). As the acceleration of the water drops so does the pressure gradient (phase 2) and although the water is still slowing the cylinder down, the deceleration is much smaller and the motion of the water itself is rather chaotic. In phase 3 the cylinder is completely submerged and sinks due to its density while on the surface the water performs complex motion. I will not be analysing phase 3, the focus of this solution is on phase 1.

3 Opening remarks

Notation:

- we are working in cylindrical coordinates (r, φ, z), with the z axis along the cylinders axis and directed downwards
- ρ_0, ρ are the densities of water and the cylinder respectively
- R, h are the radius and height of the cylinder respectively
- $\mathbf{u}(\mathbf{r}, t)$ is the fluid velocity vector field ($\mathbf{r}$ is the position vector)
- $y(t)$ is the z component of the bottom of the cylinder with respect to the surface of the water
- $a \sim b$ means that a is of the order of magnitude b
- η, σ are the viscosity and surface tension of water respectively

The following facts will be used in this solution:

- From axial symmetry: $\hat{\varphi} \cdot \mathbf{u} = 0$
- Water is incompressible (hence $\nabla \cdot \mathbf{u} = 0$)
- Euler's equation (Newton's second law for a fluid parcel)
- The dimensions of the water container are much bigger than R
- The viscosity and surface tension of water are negligible in phase 1
- the obvious $\rho > \rho_0$
- $\frac{R}{c} << \tau << \frac{R}{v_0}$

4 Phase 0 ($t < 0$)

The motion in this phase can be trivially described with $\mathbf{u}(\mathbf{r}, t) = 0$ for $t < 0$ and $y(t) = v_0 t + \frac{1}{2} g t^2$.

5 Phase 1 ($0 < t < \tau$)

This is the main part of the solution. Upon impact a fast wave travels through the water at the speed of sound c, giving it momentum. Since c is huge in comparison to any other characteristic speed (water has a high bulk modulus) we conclude $c >> R/\tau >> v_0$.

Claim: In this short time period $0 < t < \tau$ the forces of friction, surface tension and gravity: F_v, F_s, F_g have a negligible effect in comparison with the pressure.

Proof: The impulse of the pressure force is equal to the momentum given to the water in phase 1. This impulse is of the order of $I \sim \rho_0 R^3 v_0$. We also know $F_g \sim \rho R^3 g,\ F_v \sim \eta R v_0,\ F_s \sim \sigma R$ which are all $<< \rho_0 R^3 v_0 / \tau \sim F_p$ (where F_p is the pressure force on the cylinder) since τ is very small.

The motion of a parcel of water can be described by Euler's equation (pressure gradient is the dominant force as we have established):

$$\rho_0 \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p \tag{1}$$

Here $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$ is the material derivative of $\mathbf{u}$ and can be computed as:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}$$

Note that the velocity in this period is not big (it's of the order of v_0), but the acceleration is huge in comparison. Therefore we can approximate:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t}$$

We can now integrate equation (1) in time to obtain:

$$\mathbf{u}(\mathbf{r}, \tau) = \nabla \left[-\frac{1}{\rho_0} \int_0^\tau p(\mathbf{r}, t) \mathrm{d}t \right]$$

We define:

$$\phi(\mathbf{r}) = -\frac{1}{\rho_0} \int_0^\tau p(\mathbf{r}, t) \mathrm{d}t$$

Newton's second law in the z direction to the system cylinder-water:

$$\frac{dP}{dt} = \hat{\mathbf{z}} \cdot \mathbf{F}_{ext}(t)$$

Where $P = P_c + P_w$ is the total momentum in the z direction. Integrating this from 0 to τ and neglecting the external forces in this short time period we get $P(0) = P(\tau)$, conservation of momentum in the z direction before and after the impact. From this:

$$R^2 h \pi \rho (v_0 - v_1) = P_w = \Delta m v_1 = \rho_0 \int_{\text{fluid}} \mathbf{u} \cdot \hat{\mathbf{z}} \, \mathrm{d}^3 x \tag{2}$$

Here Δm is the so called added mass of the fluid. Our goal now is to find this added mass exactly. Let $\mathbf{u}'(\mathbf{r}) = \mathbf{u}(\mathbf{r}, \tau) - v_1 \hat{\mathbf{z}}$ (this is the velocity of fluid at $\mathbf{r}$ as seen by the cylinder). Consider the following boundary conditions:

1. Since the cylinder is a solid body whose velocity from its own reference frame is 0, we have $\mathbf{u}'(r < R, z = 0) \cdot \hat{\mathbf{z}} = 0$.

2. Since $p(r, z = 0, t) = 0$ for $r > R,\ 0 < t < \tau$ (this part of the fluid is in contact with the atmosphere) we have $\phi(r, z = 0) = 0$ for $r > R$. This implies $\mathbf{u} \cdot \hat{\mathbf{r}} = 0$ and also $\mathbf{u}' \cdot \hat{\mathbf{r}} = 0$ for $r > R,\ z = 0$.

3. For $|\mathbf{r}| \to \infty$ we have $\mathbf{u}' \to -v_1 \hat{\mathbf{z}}$.

Since $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} = \nabla\phi$ we have also $\nabla \cdot \mathbf{u}' = 0,\ \mathbf{u}' = -\nabla\phi'$ where $\phi' = -\phi + v_1 z$. From this we can also easily see $\nabla \times \mathbf{u}' = 0$. Now we will consider a completely different problem in electrostatics which will turn out to be analogous to this problem. Consider a dielectric ellipsoid defined by:

$$\frac{r^2}{R^2} + \frac{z^2}{c^2} \leq 1$$

Take the dielectric constant of this ellipsoid to be $\epsilon = 0$. Put this ellipsoid into a uniform electric field $\mathbf{E}_0 = -E_0\hat{\mathbf{z}}$ in vacuum. The ellipsoid develops some polarisation $\mathbf{P}$ and produces it's own field creating the total field $\mathbf{E}$. Consider the limit $c \to 0$. From planar symmetry about the plane $z = 0$ we get $\mathbf{E} \cdot \hat{\mathbf{r}} = 0$ for $r > R,\ z = 0$. For $|\mathbf{r}| \to \infty$ we have $\mathbf{E} \to -E_0\hat{\mathbf{z}}$. On the boundary between dielectrics $\epsilon\mathbf{E} \cdot \hat{\mathbf{n}}$ is continuous (see Appendix) where $\hat{\mathbf{n}}$ is the unit vector normal to the surface of separation between the dielectrics. Since we let $c \to 0$ we have $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and since $\epsilon = 0$ inside the ellipsoid we have $\mathbf{E} \cdot \hat{\mathbf{z}} = 0$ for $r < R,\ z = 0$. From Maxwell's equations we also have $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$ outside the ellipsoid. These boundary conditions are the same as conditions 1, 2 and 3 with $v_1 \leftrightarrow E_0$. We conclude that for $z \geq 0$ we have $\mathbf{u}' = \frac{v_1}{E_0}\mathbf{E}$. This a corollary of the uniqueness theorem (see Appendix). The reason for introducing this analogy is that it is easier to solve such problems using the already existing *machinery* of electrostatics.
There is a well known fact about ellipsoids in uniform fields (which is even more apparent for thin oblate spheroids) and that is that they get polarized with uniform polarisation $\mathbf{P}$ which produces its own field $\mathbf{E}'$ to make the total field $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}'$ (at every point). Since $\mathbf{P}$ is uniform inside the ellipsoid then $\mathbf{E}'$ is uniform as well, let its magnitude inside the ellipsoid be E'. To compute the added mass we turn to equation (2):

$$\Delta m v_1 = \rho_0 \int_{\text{fluid}} \mathbf{u} \cdot \hat{\mathbf{z}}\ \mathrm{d}^3x = \rho_0 \lim_{c\to 0} \int_{S_c} (\mathbf{u}' + v_1\hat{\mathbf{z}}) \cdot \hat{\mathbf{z}}\ \mathrm{d}^3x$$

Here $S_c = \{(r, \varphi, z) : \frac{r^2}{R^2} + \frac{z^2}{c^2} > 1,\ r \geq 0,\ z > 0\}$. We can now invoke our analogy (divide by v_1) and symmetry:

$$\begin{aligned}
\Delta m &= \rho_0 \frac{1}{E_0} \lim_{c\to 0} \int_{S_c} (\mathbf{E} - \mathbf{E}_0) \cdot \hat{\mathbf{z}}\ \mathrm{d}^3x = \rho_0 \frac{1}{2E_0} \lim_{c\to 0} \int_{\mathbb{R}^3 - \text{ellipsoid}} \mathbf{E}' \cdot \hat{\mathbf{z}}\ \mathrm{d}^3x = \\
&= \rho_0 \frac{1}{2E_0} \lim_{c\to 0} \left[\int_{\mathbb{R}^3} \mathbf{E}' \cdot \hat{\mathbf{z}}\ \mathrm{d}^3x - \int_{\text{ellipsoid}} \mathbf{E}' \cdot \hat{\mathbf{z}}\ \mathrm{d}^3x \right] = \rho_0 \frac{1}{2E_0} \lim_{c\to 0} \left[-\hat{\mathbf{z}} \cdot \int_{\partial\mathbb{R}^3} V' \mathrm{d}\mathbf{a} + \frac{4}{3}\pi R^2 c E' \right]
\end{aligned}$$

$$\implies \Delta m = \lim_{c\to 0} \frac{2}{3}\pi R^2 c \rho_0 \frac{E'}{E_0} \tag{3}$$

Here we have used that the volume of the ellipsoid is $\frac{4}{3}\pi R^2 c$ and that the field inside it is uniform, as well as that $\mathbf{E}' = -\nabla V'$. Thus to compute the added mass we need only to calculate the electric field inside the ellipsoid. On the surface $\mathcal{S}$ of the ellipsoid there is a bound surface charge $\sigma_b = \hat{\mathbf{n}} \cdot \mathbf{P}$ where $\mathbf{P} = -(\epsilon - \epsilon_0)(E' + E_0)\hat{\mathbf{z}} = \epsilon_0(E' + E_0)\hat{\mathbf{z}}$ is the polarisation (since $\epsilon = 0$). Let us parameterize $\mathcal{S}$ as $r = R\sin\theta,\ z = c\cos\theta$ for $\theta \in [0, \pi\rangle$. Let $x << c^2/R^2$ be some small real number. Now we consider the straight line path $(0, 0) \to (0, xc)$. Obviously we know:

$$E'xc = -\int_{(0,0)}^{(0,xc)} \mathbf{E}' \cdot \mathrm{d}\mathbf{l} = V'(0, xc) - V'(0, 0)$$

We will pick the zero of the potential V' at infinity. Note that since $\hat{\mathbf{r}} \cdot \mathbf{E}' = 0$ for $z = 0$ we have that $V'(0, 0) = 0$. Also we can find $V'(0, xc)$ by evaluating the Coulomb integral of σ_b:

$$V'(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}a' \tag{4}$$

We will now evaluate the integral (4) for the point $\mathbf{r} = (0, xc)$. Let $\alpha = \frac{c}{R}$. Then the vector $\mathrm{d}\mathbf{l}'$ along the surface of the ellipsoid is:

$$\mathrm{d}\mathbf{l}' = \mathrm{d}r'\hat{\mathbf{r}} + \mathrm{d}z'\hat{\mathbf{r}} = R(\cos\theta\hat{\mathbf{r}} - \alpha\sin\theta\hat{\mathbf{z}})\mathrm{d}\theta$$

Then we have:

$$\mathrm{d}a' = 2\pi r' \mathrm{d}l' = 2\pi R^2 \sin\theta\sqrt{\cos^2\theta + \alpha^2\sin^2\theta}\,\mathrm{d}\theta$$

Note that $\hat{\mathbf{n}}$ is just the unit vector perpendicular to $\mathrm{d}\mathbf{l}'$:

$$\hat{\mathbf{n}} = \frac{\alpha\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\mathbf{z}}}{\sqrt{\cos^2\theta + \alpha^2\sin^2\theta}}$$

And thus:

$$\sigma_b(\theta) = \mathbf{P}\cdot\hat{\mathbf{n}} = \epsilon_0(E' + E_0)\frac{\cos\theta}{\sqrt{\cos^2\theta + \alpha^2\sin^2\theta}}$$

Also $|\mathbf{r} - \mathbf{r}'| = R\sqrt{\sin^2\theta + \alpha^2(\cos\theta - x)^2}$. Substituting into (4) and using $x << \alpha^2$ and a substitution $u = \cot\theta$:

$$V'(0, xc) = \frac{1}{4\pi\epsilon_0}\int_0^\pi \frac{1}{R\sqrt{\sin^2\theta + \alpha^2(\cos\theta - x)^2}}\epsilon_0(E' + E_0)\frac{\cos\theta}{\sqrt{\cos^2\theta + \alpha^2\sin^2\theta}}2\pi R^2\sin\theta\sqrt{\cos^2\theta + \alpha^2\sin^2\theta}\mathrm{d}\theta$$

$$= \frac{1}{2}(E' + E_0)R\int_0^\pi \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \alpha^2\cos^2\theta - 2\alpha^2\cos\theta x}}\mathrm{d}\theta = \frac{1}{2}(E' + E_0)R\int_0^\pi \frac{\cos\theta}{\sqrt{1 + \alpha^2\cot^2\theta}}\left[1 + \frac{x\alpha^2\cos\theta}{\sin^2\theta + \alpha^2\cos^2\theta}\right]\mathrm{d}\theta$$

$$= \frac{1}{2}(E' + E_0)x\alpha^2\int_0^\pi \frac{\cos^2\theta}{\sin^2\theta(1 + \alpha^2\cot^2\theta)^{3/2}}\mathrm{d}\theta = (E' + E_0)Rx\alpha^2\int_0^\infty \frac{u^2\ \mathrm{d}u}{(1 + u^2)(1 + \alpha^2u^2)^{3/2}}$$

Now to evaluate the last integral we will use the Leibniz rule. Let us define:

$$I(b) = \int_0^\infty \frac{(u^2 + b)\ \mathrm{d}u}{(1 + u^2)(1 + \alpha^2u^2)^{3/2}},\ b \in \mathbb{R}$$

Note that:

$$I(1) = \int_0^\infty \frac{\mathrm{d}u}{(1 + \alpha^2u^2)^{3/2}} = \frac{1}{\alpha}\int_0^{\pi/2}\cos\theta\mathrm{d}\theta = \frac{1}{\alpha},\ (\text{Here: } \alpha u = \tan\theta)$$

Also we can use $\alpha << 1$ and differentiation under the integral sign:

$$I'(b) = \int_0^\infty \frac{\mathrm{d}u}{(1 + u^2)(1 + \alpha^2u^2)^{3/2}} = \int_0^\infty \frac{\mathrm{d}u}{1 + u^2} = \frac{\pi}{2}$$

Integrating this:

$$I(b) = I(1) + \frac{\pi}{2}(b - 1) \Longrightarrow I(0) = \int_0^\infty \frac{u^2\ \mathrm{d}u}{(1 + u^2)(1 + \alpha^2u^2)^{3/2}} = \frac{1}{\alpha} - \frac{\pi}{2}$$

Now we can put all this together:

$$xcE' = V'(0, xc) = (E' + E_0)Rx\alpha^2\left(\frac{1}{\alpha} - \frac{\pi}{2}\right)$$

Note that we have $\alpha << 1$ and thus:

$$E' = \frac{2}{\pi}\frac{1}{\alpha}E_0$$

Substituting into (3) we get:

$$\Delta m = \frac{4}{3}\rho_0 R^3 \tag{5}$$

6 Phases 2 and 3 ($t > 0$)

The phases 2 and 3 are chaotic. The cylinder still has some added mass, although it would be very difficult to find. In the beginning of phase 2, $y(t)$ can be approximated as $y(t) = v_1 t$ for small times $t > 0$.

7 Measurements

From equations (2) and (5) one can solve for ρ:

$$\boxed{\rho = \frac{4}{3\pi}\frac{v_1}{v_0 - v_1}\frac{R}{h}\rho_0}$$

Since we know $\rho_0 = 1$ g cm^{-3}, to find ρ we need only measure the ratios $\frac{v_1}{v_0-v_1}$, $\frac{R}{h}$. This is done by pixel and frame counting in the video. We can use a pixel ruler to measure R and h in pixels and also $y(t)$ in various frames just before and after the impact. By finding best fit lines for $y(t)$ before and after the impact we can determine v_0, v_1 in units of pixels/frame. For more precise measurements, rather than measuring y it is more convenient to measure Y, the distance from the top side of the cylinder from the top of the video frame. The data is presented in Table 1.

t/frames	Y/pixels
-7	58
-6	73
-5	89
-4	108
-3	122
-2	141
-1	157
0	175
1	188
2	200
3	212

Table 1: $Y(t)$ directly before and after the impact

From this data $v_0 = 16.7$ pixels/frame and $v_1 = 12$ pixels/frame are computed (using Microsoft Excel). Also $R = 180$ pixels and $h = 140$ pixels. These values can then be substituted in the boxed formula to give:

$$\boxed{\rho = 1.39 \text{ g cm}^{-3}}$$

Note that these measurements are not very precise and thus the real density may be a bit different.

8 Appendix: the uniqueness theorem

Theorem: If a potential ϕ in a given volume V satisfies the Laplace equation and is also specified on the boundary ∂V then ϕ can be uniquely determined.
Proof: Suppose there exist two such potentials ϕ_1, ϕ_2, we have $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$ for $\mathbf{r}$ in V and also $\phi_1 = \phi_2$ for $\mathbf{r}$ in ∂V. Let $\phi_3 = \phi_1 - \phi_2$. We have $\phi_3 = 0$ for $\mathbf{r}$ in ∂V and $\nabla^2\phi_3 = 0$, from which we know that all the extreme values of ϕ_3 are achieved on the boundary but that means $0 \leq \phi_3(\mathbf{r}) \leq 0$ for all $\mathbf{r}$ in V so $\phi_3 = 0$ and $\phi_1 = \phi_2$ $\blacksquare$.