# PC 2024 task 1

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## 1 Some assumptions and observations

**Before we start solving this problem, we need to simplify it as much as possible due to hydrodynamics being extremely complicated. We shall take the water as an incompressible fluid. We shall ignore the effects of the air above the water on the cylinder. We shall consider the cylinder to be a rigid body.**

**Since many of the more complex mechanisms of fluids begin happening only after some time passes, we can consider the moments right before and right after the collision of the cylinder with the water. Due to the small time interval we may consider it as a plastic collision between 2 objects. To do so, we need to use a term known as added mass which provides an easy way to see how a body acts inside a liquid, pretending it just has a higher mass due to it needing to add momentum to the liquid surrounding it as well as to itself. Since we are only considering the moment right after the collision, we can, instead of considering the whole cylinder, just consider the part immersed in the water. Here that is a very thin disc, which we shall approximate to be infinitely thin. Added mass is defined as follows:**

$$\int \frac{\rho v_{rel}^2}{2} dV = \frac{m_{add} v_{cm}^2}{2}$$

**Where $\rho$ is the density of the liquid, here the liquid being water, $v_{rel}$ the velocity of the water relative to the body and $v_{cm}$ the velocity of the center of mass of the body. Here we have no rotation and the body is rigid so it is the speed of the body itself. The volume integral is over the whole volume of water. Due to this generally being used when bodies are fully immersed, the integral is over the whole of space.**

**Regarding measurements, we can only measure relative changes, so we shall aim to find a solution which can be expressed in only measurable ratios and well known constants.**

## 2 Calculating the added mass

**To calculate the added mass of the thin disc in the water, we shall use an analogy with electromagnetic fields. We can consider the disc as the limit of an ellipsoid of sides, R, R and d as the ratio $\frac{d}{R}$ tends to 0. The reason for considering an ellipsoid is due to ellipsoids having constant magnetic fields when their magnetization is constant. To go through with the analogy we take the ellipsoid to be superconductive, that is $\mu_r = 0$, that is in it the net magnetic field is 0, similar to, how in the case of our cylinder, no water flows through it. We may draw a parallel between magnetic and velocity fields because their governing equations in this case are the same. We consider water to be basically incompressible, that is $\nabla \cdot \vec{v} = 0$. According to Gauss' law $\nabla \cdot \vec{B} = 0$ as well. Since the water is in a resting state at the beginning moment, and since there is no torque acting on the water/body, we can easily infer that $\nabla \times \vec{v} = 0$ in the moments we are considering the system. Similarly to that, since we shall be considering a magnetostatic system, $\nabla \times \vec{B} = 0$ as well since there will be no currents outside the body and we are comparing the flow of water with the magnetic field outside the ellipsoid. Aside from that, the boundary equations are also the same. Due to our ellipsoid being superconductive we can see that inside of it, $B$ is equal to 0, just like how the water's velocity is 0 inside of the cyllinder. Furthermore, the magnetic field on the boundary is fully tangential, just like the no slip condition necessitates in hydrodynamics. Because of this we can see that the governing equations are the same. To convert between the 2 we need to set an arbitrary measure with which we can make an analogy. Here we have chosen to equate the energies of the kinetic energy of water and the magnetic field in the air in our analogous system.**

$$\int_\omega \frac{\vec{B}^2}{2\mu_0} dV = \int_\omega \frac{\rho \vec{v}^2}{2} dV$$
$$\frac{\vec{B}^2}{\mu_0} = \rho \vec{v}^2$$
$$\vec{v} = \frac{\vec{B}}{\sqrt{\rho \mu_0}}$$

**First let's define the coordinate system in which we shall work with the ellipsoid. We shall use a cylindrical coordinate system $(r, \theta, z)$, where the point $(0, 0, 0)$ is in the center of the ellipsoid and the $z$ axis aligns with the short axis of the ellipsoid. Or in the other words the ellipsoid's equation is $\frac{r^2}{R^2} + \frac{z^2}{d^2} = 1$**

**In the same way we can consider the motion of a rigid body of constant velocity in a liquid as the body resting and the liquid moving towards it with the same velocity, we shall put our ellipsoid into a magnetic field of strength $\vec{B_0} = B_0 \hat{z}$. Since we want our magnetization to reduce the field to 0 inside of the ellipsoid, it's clear that $\vec{M} = -M\hat{z}$. To get**

**the current distribution in the ellipsoid due to magnetization, we can use the following 2 formulas to obtain the volume, that is surface current distributions:**

$$\vec{J}_b = \frac{d\vec{I}}{dA} = \boldsymbol{\nabla} \times \vec{M} = 0$$
$$\vec{K}_b = \frac{d\vec{I}}{dl} = \vec{M} \times \hat{n}$$

**The first equation is trivially seen to be 0. The** $\hat{n}$ **in the 2nd equation is the normalized normal vector to the surface of a body. To find it for our body we can first see that due to its radial symmetry it is enough to consider just the z-r plane. Considering the upper half first, we can take that** $z = \sqrt{d^2 - \frac{d^2 r^2}{R^2}}$**, thus the tangential vector can be written as** $\hat{t} = cos\alpha\hat{r} + sin\alpha\hat{z}$**, where** $tg\alpha = \frac{dz}{dr}$**. Since the normal vector is perpendicular to it, while still being in the same plane, it is either of the form** $\hat{n} = sin\alpha\hat{r} - cos\alpha\hat{z}$ **or** $\hat{n} = -sin\alpha\hat{r} + cos\alpha\hat{z}$**. Due to the geometry of the problem we can see that we want** $\hat{r}$ **to be positive both times. Through trigonometry and that condition we can find that** $sin\alpha = \frac{tg\alpha}{\sqrt{1+tg^2\alpha}}$**. From this we obtain:**

$$tg\alpha = -\frac{d^2 r}{\sqrt{d^2R^4 - d^2r^2R^2}}$$
$$sin\alpha = -\frac{d^2 r}{\sqrt{d^2R^4 + d^4r^2 - d^2r^2R^2}} = -\frac{d^2 r}{\sqrt{d^2R^4 - d^2r^2R^2}}$$

**We shall use the 2nd formula for** $\hat{n}$**,** $cos$ **is obviously positive. Here we ignored the quartic** $d$ **term due to** $\frac{d}{R} << 1$**.**

$$\vec{M} \times \hat{n} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & 0 & -M \\ sin\alpha & 0 & cos\alpha \end{vmatrix} = -Msin\alpha\hat{\theta} = \frac{d\vec{I}}{dl}$$

**Since** $sin\alpha$ **would have to be positive even if** $d$ **were negative, we can see that this relation holds true even for the bottom half of the ellipsoid. We can easily see that this solution corresponds to circular loops around the ellipsoid. With that we can easily calculate the magnetic field in the center of the ellipsoid due to currents, which should be equal to** $B_0$ **but in the opposite direction. Due to** $d$ **being small, in the limit we are considering we shall consider the center of the ellipse to be approximately in the same plane as the circular loops. Then the 2 distinct loops at the edges converge into 1 as well. We will be going from the center to the edge radially, so** $dl$ **turns to** $dr$**.**

$$dB = \frac{\mu_0 dI}{r} \; dB = \frac{\mu_0 M sin\alpha dr}{r} = \frac{\mu_0 M d^2 dr}{\sqrt{d^2R^4 - d^2r^2R^2}}$$
$$B_0 = \mu_0 M d \int_0^R \frac{1}{\sqrt{R^4 - r^2R^2}} dr = \mu_0 M d \frac{\pi}{2R}$$

**Of course, this magnetic field is in the $-\hat{z}$ direction to counteract the original $B_0$.**

**To find the added mass of the disc, we continue the analogy and see that the magnetic field from the ellipsoid's currents, $B_s$, corresponds to the relative velocity of water to the disc. Since we decided that energy is equal in our analogy, we can write this down as the following:**

$$\int \rho v_{rel}^2 dV = \int \frac{B_s^2}{\mu_0} dV = m_{add} v_{cm}^2$$

**$v_{cm}$ is the velocity of the center of mass of the disc/cylinder in a frame where the water is initially at rest, it corresponds to $B_0$. One way to obtain the solution to this integral is by seeing the work done by putting the ellipsoid in the magnetic field $B_0$. Since the ellipsoid has constant magnetization we can imagine it as a lot of dipoles $m = M\frac{4\pi R^2 d}{3}$. Before it was put into the field, they had random orientation, while after they are all oriented in the $-\hat{z}$ direction. In other words, work had to be put in to turn them around. The magnetic field inside the ellipsoid, as shown earlier, is proportional to $M$, that is to say to $m$. The energy at the start inside the ellipsoid is 0.**

$$U = -\vec{m} \cdot \vec{B}$$
$$W = \int_0^{B_0} m dB = \frac{mB_0}{2}$$
$$\Delta E_p = W = \int \frac{(\vec{B_s} + \vec{B_0})^2}{2\mu_0} dV - \int \frac{B_0^2}{2\mu_0} dV =$$
$$\int \frac{B_0^2}{2\mu_0} dV + \int \frac{\vec{B_0} \cdot \vec{B_s}}{\mu_0} dV + \int \frac{B_s^2}{2\mu_0} dV - \int \frac{B_0^2}{2\mu_0} dV$$
$$W = B_0 \int \frac{\hat{z} \cdot \vec{B_s}}{\mu_0} dV + \int \frac{B_s^2}{2\mu_0} dV$$

**The solution to the first integral may be easily obtained since the magnetic field lines of $B_s$ are all closed, with the exception of the 1 going through the middle of the ellipsoid, but that one occupies an infinitesimally small amount of space so it does not influence the result. Since the lines are closed, every time you have some value of $B\hat{z}$ in the magnetic field line, you also have a value of $-B\hat{z}$ due to it needing to be a closed loop. Thus the result of that integral is 0. Using this and our previous equations we can obtain the result for the added mass:**

$$m_{add} v_{cm}^2 = 2W = mB_0 = MB_0 \frac{4\pi R^2 d}{3} = B_0^2 \frac{4\pi R^2 d}{3} \frac{2R}{\pi d \mu_0}$$
$$\text{Using our analogy where: } B_0 = v_{cm}\sqrt{\rho\mu_0}$$
$$m_{add} v_{cm}^2 = v_{cm}^2 \frac{8R^3\rho}{3} \quad m_{add} = \frac{8R^3\rho}{3}$$

**This result aligns with various literature. Though we need to be vary of the fact that only half of the space we consider is actually occupied with water, due to the symmetrical nature of magnetic equations, it's**

easy to see that if we set that the top half of the space contributes 0 energy, we get a number twice as small. Air has approximately the same effect on the system as that, so we may consider its energy change due to the cyllinder to be 0. So we shall use $m_{add} = \frac{4R^3\rho}{3}$

# 3 Obtaining the density using measurable quantities

Now that we have the added mass, we can simply consider the system of water + the cylinder, due to us considering a very short time frame, we can approximate that no outer forces are acting upon the system (the only one being gravity which is able to be ignored in this short of a time frame) so the momentum is conserved. We shall label the velocity of the cylinder right before the collision as $v_0$ and right after as $v_1$, due to the speeds being obviously significantly slower than the speed of sound in water, we may consider the momentum transfer in water to be effectively instant. Due to the short time frame and slow speed, little energy is lost due to resulting shockwaves. $M$ is the mass of the cylinder, $\rho_c$ its density, while $\rho$ is the density of water which is well known to be about 1000 kg/$m^3$. The added momentum from the water is very easily obtained from its added mass. $R$ and $h$ are the radius and height of the cylinder respectively.

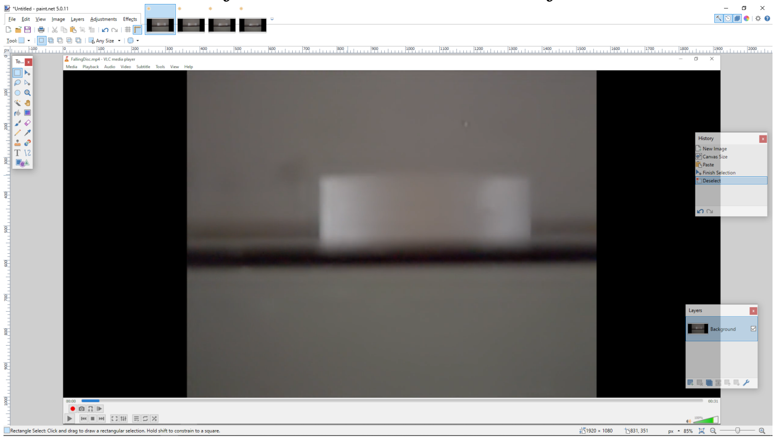
$$Mv_0 = Mv_1 + p_{add}$$
$$\frac{m_{add}v_{cm}^2}{2} = \frac{p_{add}v_{cm}}{2}$$
$$p_{add} = m_{add}v_{cm}$$
$$Mv_0 = Mv_1 + m_{add}v_1$$
$$\rho_c hR^2\pi v_0 = \rho_c hR^2\pi v_1 + m_{add}v_1$$
$$\rho_c = \frac{\frac{4R^3\rho}{3}v_1}{hR^2\pi(v_0-v_1)} = \frac{\frac{4R\rho}{3}v_1}{h\pi(v_0-v_1)}$$
$$\rho_c = \frac{4\rho}{3\pi}\frac{R}{h}\frac{1}{(\frac{v_0}{v_1}-1)}$$

The quantities $\frac{R}{h}$ and $\frac{v_0}{v_1}$ are measurable from the given video without needing any reference frame.

# 4 Measurements from the video

To measure the speed of the cylinder, we put the video in a video viewing tool (VLC) and go frame by frame and screenshot each one. Then we put the screenshot in an image editing tool (paint.net) where you can place the mouse cursor approximately near the top of the cylinder (I considered the top the part where it stops being blurry) and write down the pixel coordinates given by the programme. All the screenshots were taken in the same way, so the coordinates are absolute, given that the camera is steady (which we shall assume is the case). We shall also ignore the effects of different viewing angles from the camera due to us not having enough information about its whereabouts. Aside from that, the cylinder seems to have constant height throughout the video, so we can say that the effects from the camera's physical placement are irrelevant. (We shall confirm this when we measure the cylinder's height).

We chose the frame of impact as a "frame 0" and measured everything relative to that. We only went a few frames forward/backwards so the effects of gravity/the cylinder going deeper are able to be ignored. Below this we have an image from the image editing software, where the coordinates are visible in the bottom right (the coordinates are measured such that the upper left corner is (0,0) and the coordinates grow going to the bottom right. The 2nd coordinate is the y axis which is the only relevant one for velocity.



For calculations we used Excel due to it being convenient. Along the way we also calculated the standard deviations for the velocities, which turned out small so we can say that we took a small enough time frame to ignore the effects of gravity and the likes.

| FRAME | px | | FRAME JUMP | | px/frame | | Stdev v0 | Stdev v1 |
|---|---|---|---|---|---|---|---|---|
| -4 | 232 | | From -4 to -3 | | 31 | | 0.829156 | 0.471405 |
| -3 | 263 | | From -3 to -2 | | 29 | | | |
| -2 | 292 | | From -2 to -1 | | 30 | | | |
| -1 | 322 | | From -1 to 0 | | 29 | | | |
| 0 | 351 | | From 0 to 1 | | 22 | | | |
| 1 | 373 | | From 1 to 2 | | 22 | | | |
| 2 | 395 | | From 2 to 3 | | 21 | | | |
| 3 | 416 | | | | | | | |

| FRAME | Left | Right | Top | Bottom | Radius | Height | R/h |
|---|---|---|---|---|---|---|---|
| First | 740 | 1364 | 112 | 374 | 312 | 262 | 1.19084 |
| Second | 745 | 1361 | 263 | 523 | 308 | 260 | 1.184615 |

**We can see that in our chosen time frame the velocities are relatively constant (red indicates $v_0$, while green indicates $v_1$) and with the small standard deviation we can say with high confidence that our measurements for the velocities are good. To make our ratio $\frac{v_0}{v_1}$ more accurate we shall say that $v_0 = \frac{119}{4} px/frame$ and $v_1 = \frac{65}{3} px/frame$.**

**All the measurements for the radius and height are taken in pixels. The first frame is a random frame when the cylinder just fully falls into view, while the second frame is frame -3 due to it having a good view of the cylinder. These 2 frames were picked due to them being quite far apart but still above the water where refraction might cause an issue. We can see that the cylinder height and radius are roughly the same, so the physical effects from the camera's placement are able to be ignored. for the ratio $\frac{R}{h}$ we shall use the arithmetic average of the 2 calculated ratios. $\frac{R}{h} = \frac{156}{262} + \frac{154}{260}$ As a reminder, we took $\rho = 1000 kg/m^3$**

$$\rho_c = \frac{4\rho}{3\pi} \frac{R}{h} \frac{1}{(\frac{v_0}{v_1} - 1)} \quad \rho_c = 1351.16 kg/m^3$$

**Of course, here were many opportunities to make a mistake due to having a quite imprecise measuring tool so this result is by no means accurate, but we can infer that the density should be somewhere between 1300 $kg/m^3$ and 1400 $kg/m^3$.**