Physics Cup -24' Problem 1 version 2.0

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1 General considerations.

In our problem we need to first consider that the only place in the video we can comfortably take measurements in is the 2 frames right before and right after the impact as after that the motion of the water becomes to complicated for us to measure. During the impact the water is also displaced, do to this the water will begin to move too, now that displaced fluid can be considered as having some kinetic energy, since the energy is proportional to the mass and velocity squared we can conclude from this that the fluid will have some added mass. We also need to consider that during the impact with the water right in that moment the velocity of the water will be for that brief moment the same as the one of the cylinder as in the collision will be plastic.Now all that's left is to consider some conservation law, since the displacement of the cylinder between two frames is small the acceleration do to gravity is negligible, we can considered that we can use the conservation of momentum do to the impact.

2 Calculation of the added mass.

Since it's required to know advanced math such as solving the Laplance differential equation to obtain an expression for the added mass (a tool that is way outside of the IPhO Syllabus) and other advanced tools to calculate the added mass of the cylinder when its hitting the surface of the water we will instead as provided in the hint calculate it using the analogy of magnetic fields and velocity fields in water.
First its convenient to prove said relation: We know that for an incompressible fluid $\rho = const.$ we have an equation for the velocity field vector:

$$\nabla . \vec{v} = 0$$

as well as the fact its vortex free:

$$\nabla \times \vec{v} = 0$$

Now let's first consider the Gauss law for magnetic fields:

$$\nabla . \vec{B} = 0$$

now in the case of magnetic fields the Ampere's law says:

$$\nabla \times \vec{B} = \mu_o \vec{J} + \epsilon_o \mu_o \frac{\partial \vec{E}}{\partial t}$$

but in our case since there are no electric fields changing in time as well as we consider the fact that vector $\vec{J}$ far away can be taken to be 0 we get:

$$\nabla \times \vec{B} = 0$$

from here we can see that both of these equations have the same relationship. We can denote the relation for simplicity: $\vec{v} = c\vec{B}$ (where c is a constant) from here we will simply write the relation of the energy:

$$\frac{1}{2} \rho V v^2 = \frac{1}{2\mu_o} B^2 V$$

which gives us our desired constant $c = \sqrt{\frac{1}{\mu_0 \rho}}$ and proves the relation.

in our considerations we will study a superconductor in homogeneous magnetic field B_o.

instead of considering a cylinder diving into the water lets consider instead an ellipsoid of some unknown polarization being put into an a homogeneous magnetic field B_o.

Since the magnetisation field M is uniform inside the ellipsoid the magnetic moment can be considered as: $m = MV = \frac{4\pi}{3}abc$ where a, b, c are its semi axis, in our case since we will use them for calculation we can assume that will eventually need to consider a body that is symmetric (infinitely thin plate of radius R) whose circular semi-axis length is always R such that a=b=c=R so we can write $m = \frac{4}{3}\pi R^3 M$.

Do to the fact that the ellipsoid is a superconductor and put inside a homogeneous magnetic field B_o the magnetic filed inside the body in the reference frame matched with the rest frame will be 0, and on the other side in the frame of reference of the body it will be $B_o + B_s$ that is because of the fact that the currents have redistributed themselves on the surface such that they create an opposite magnetic field to cancel the external magnetic field.

Now we need to calculate m in terms of B_o. Let's consider an ellipse stretched along one of its axis, let's also assume that the body also has some kind of small thickness $d(r)$ away from the axis. We can begin our calculations using the The Ampère's circuital law in the middle of the ellipse $\oint_C Bd\ell = \mu_0 I_C = \iint_S J.dS$ this tells us :

$$B_o R = \int \mu_o J.dS$$

Where we can consider only one side of the ellipse since it will cancel out do to the symmetry of the body, now we have some ellipsoid shape with an infinitely small thickness d(r) such that $R >> d(r)$, the surface current do to the body being of some uniform magnetisation will have its current redistributed such that $J = J_o cos(\alpha)$ away from the axis, using the hint we have an expression for dS and since the polarization is constant over the entire surface of the body we get for magnetization $M = J_o d_o$.

$$dS = d_o dl$$

where dl is the small element in the y axis as oppose to our R that is the semi-axis length of the other axis. from here for our integral we get after substituting for $cos\alpha$ and dS:

$$B_o = \frac{\mu_o J_o}{R} \int d_0 \frac{\sqrt{R^2 - l^2}}{R} dl$$

$$B_o = \frac{\mu_o M}{R} \int \sqrt{1 - \frac{l^2}{R^2}} dl$$

we can substitute now $x = \frac{l}{R}$ this changes the integration boundaries from -R to R (they were like this since we assumed each axis to be of same length and our calculations started in the middle of the body) to from -1 to 1. now we have:

$$B_o = \mu_o M \int_{-1}^{1} \sqrt{1 - x^2} dx$$

calculating the integral $\int_{-1}^{1} \sqrt{1 - x^2} dx$ gives us $\frac{\pi}{2}$.

now substituting for $m = \frac{4}{3}\pi R^3 M$ we obtain:

$$B_0 = \frac{3\mu_o m}{8R^3}$$

so $m = \frac{8B_o}{3\mu_o} R^3$.

Now we need to consider three type of energy changes the change of energy of the field inside the ellipsoid body ΔU_i, the change of energy of the region not covered by the body ΔU_o and the change in potential energy of the magnetic dipole ΔU_p. This gives us from the conservation law:

$$\Delta U_i + \Delta U_o + \Delta U_p = 0$$

Let's also remember that we considered the ellipsoid as an infinitely thin body such that far away its volume can be considered $V = 0$. Now since the volume of our body is 0 the energy conservation law for our considered problem gives us:

$$\frac{1}{2} m B_o = \frac{1}{2\mu_o} \int B_s^2 . dV$$

Knowing that in our considered problem we have water moving with the same velocity as the cylinder during our plastic collision, we know that the magnetic field inside our body has to match to the magnetic field right outside the ellipsoid, now we can simply use the fact that outside of the ellipsoid the magnetic field and electric field are equivalent as long as we consider points outside the dipoles, using Gauss law and for a moment switching from magnetic to electric fields we get that:

$$E_s(r) = E_o(\frac{R}{r})^n <=> r \geq R$$

This also matches with our case for the velocity fields since the impact in our case is plastic and the velocity of the cylinder will have to be the same as the one of the body of water moving along with it.Now switching back to magnetic fields we see that right outside of the ellipsoid the magnetic field is $|B_o| = |B_s(R)|$.

Substituting for m as well as the constant c derived earlier we get that:

$$\frac{1}{2}(\frac{8}{3}R^3)\frac{B_o^2}{\mu_o} = \frac{1}{2\mu_o}\int B_s(R)^2.dV$$

$$m_a = \frac{8}{3}\rho R^3$$

which is the added mass of a infinitely thin circular flat disc, but in the case of our considered problem the disc is only partially socked in water, do to this fact we must imagine how will our added mass change in such a case, we easily that in the case of a disc lifting up from the water (when only absolutely downward part is inside the water) as well as it case of diving in the water will be the same (since its basically the same thing just reversed), we also know that when the body of infinitely thin disc is fully submerged the added mass will be the sum of both the point when the disc is lifting from the water as well as it splashing into it, both the added masses in case of it diving and lifting up from the water have to be the same since its essentially the same plate so $m_a = \frac{8}{3}\rho R^3 = m_s + \Delta m_a = 2\Delta m_a$:

$$\Delta m_a = \frac{4}{3}\rho R^3$$

This will be the added mass of the disc diving into the water right at the moment of the impact.

3 Measurements and the numerical solution for the density of the cylinder.

Since our measurements will be based on the difference in frames the change in the acceleration of the body do to gravity will be negligible, from this we can consider some sort of conservation law.
we can't use energy do to the fact that as provided in the hint the shock wave will disperse some energy but we can use the conservation of momentum in our problem.

Now our equation of momentum gives us:

$$M\vec{v}_o = (M + \Delta m_a)\vec{v}_1$$

$$\rho_c \pi R^2 L v_o = (\rho_c \pi R^2 L + \frac{4}{3}\rho R^3)v_1$$

after rearranging for our equation we obtain for our expression:

$$\frac{\rho_c}{\rho} = \frac{4}{3\pi(\frac{v_o}{v_1} - 1)}(\frac{R}{L})$$

where ρ_c is the density of the cylinder and ρ is the density of the water.

now all that is left is to find the value of $\frac{v_o}{v_1} = k$ as well as to find the value of $\frac{R}{L}$ experimentally.

since the task is a semi-experimental one we have to take into the account that the answer will need to have some error analysis.
To find all the values precisely I have used `https://physlets.org/tracker/` for easier measurements and an a ability to mark the spots in the image with points in the next frame much more easily and to get more precise answer, the frame was taken with an a in app ruler, I drew a line in the middle of the cylinder taken the point mass option, retraced the image and found the length of difference between the top.

We can find easily the ratio between the radius and the length of the cylinder from the video using just a ruler.
We get:

$$R = (4,54 \pm 0,01)(units)$$

$$L = (3,47 \pm 0,01)(units)$$

since we take ratios the units will cancel out. Although we need to remember that the values are approximates since the picture of the cylinder is not very sharp.

Our measurements will be taken using the frame right before and right after the impact since the latter motion of the fluid becomes to complicated for us to find an exact solution.

Now, to find the value of $\frac{v_o}{v_1}$ is a bit trickier, we have to use the fact that the first frame will be the moment right when it hits the water, this will give us the velocity v_1, the exact frame before that will tell us the velocity before the impact v_o, we can copy a small piece of the cylinder (its top) and compare it to the next frame we will mark those points on the picture with the light blue dot, than we measure the length using the ruler giving us a velocity in arbitrary units. The results from the experiment are shown in the small table below.

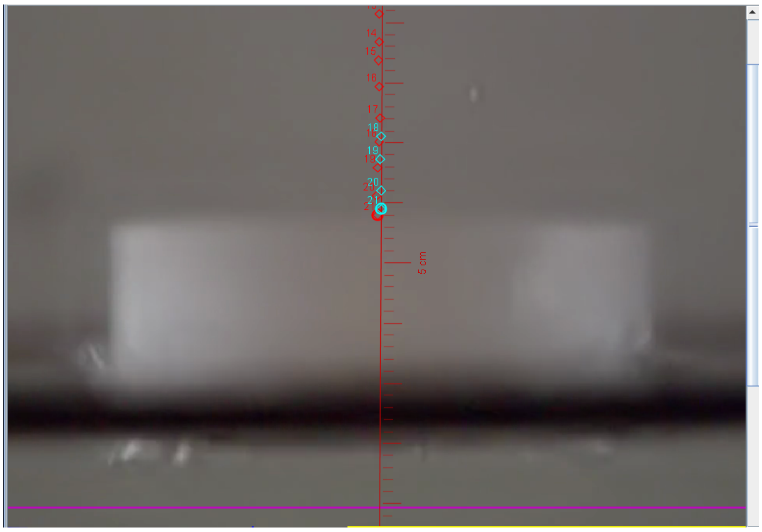

Figure 1: The exact frame taken in the measurement which tells us the value of v_1 (the light blue with number 21)

t (s)	x (cm)	y (cm)	v_y (cm/s)
0.720	10.16	6.168	
0.760	10.15	5.800	-11.17
0.800	10.15	5.274	-10.42
0.840	10.15	4.966	-6.609
0.880	10.14	4.745	

Figure 2: the table for the values of v_1 and v_o (the one showcasing the value when it dives into water is marked with the blue color.)

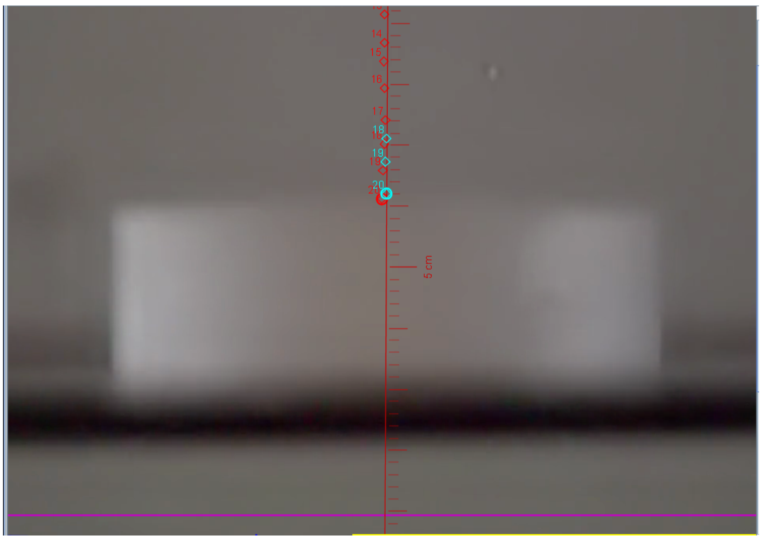

Figure 3: the frame right before the impact took place (v_o)

As before the units will cancel out, We get for our velocities:

$$v_o = 10,420 \pm 0,005 \frac{units}{frame}$$

$$v_1 = 6,609 \pm 0,005 \frac{units}{frame}$$

From here all that is left is to do error analysis, we have $f(R, L, v_o, v_1) = \frac{4}{3\pi(\frac{v_o}{v_1}-1)}(\frac{R}{L})$

$$\Delta u_o = \sqrt{(\frac{\partial f}{\partial v_1})^2 \Delta v_1^2 + (\frac{\partial f}{\partial R})^2 \Delta R^2 + (\frac{\partial f}{\partial L})^2 \Delta L^2 + (\frac{\partial f}{\partial v_o})^2 \Delta v_o^2}$$

taking the partial derivatives:

$$\Delta u_o = \sqrt{(\frac{4v_0R}{3L\pi(v_o - v_1)^2})^2 \Delta v_1^2 + (\frac{4}{3\pi(\frac{v_o}{v_1} - 1)}(\frac{1}{L}))^2 \Delta R^2 + (\frac{4}{3\pi(\frac{v_o}{v_1} - 1)}(\frac{R}{L^2}))^2 \Delta L^2 + (-\frac{4v_0R}{3L\pi(v_o - v_1)})^2 \Delta v_o^2}$$

Substituting in the values:

$$\Delta u_o \approx 0,00565$$

This gives us for $\frac{\rho_c}{\rho}$

$$\frac{\rho_c}{\rho} \approx (0,96297 \pm 0,00565)$$

Let's assume that the density of the water is $1000\frac{kg}{m^3}$ So finally we get for our density of cylinder:

$$\rho_c \approx (962,97 \pm 5,65)\frac{kg}{m^3}$$

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Now we must also remember that the final answer might deviate from the actual value, that's mostly because the image is not the sharpest.

Calculation of the integral $\int_{-1}^{1} \sqrt{1-x^2}dx$:

$$x = sin\theta, dx = cos\theta d\theta$$

$$\int \sqrt{1 - sin^2\theta} cos\theta d\theta = \int cos^2\theta d\theta$$

Now knowing that $cos^2x - sin^2x = cos2x$ we have:

$$\int \frac{1}{2} + \frac{cos2\theta}{2} d\theta$$

so we have:

$$\frac{1}{2}arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + C$$

plugging in the limits from -1 to 1 we have:

$$\frac{1}{2}(arcsin(1) - arcsin(-1)) = \frac{1}{2}\pi$$