PhysicsCup 24 Problem 1

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1 Added mass of a thin disk of radius R

In order to solve this problem we will follow the given hints. Hence we first calculate the added mass of a thin disk with radius R. As given in the hint, we calculate the magnetic polarizability of a superconducting ellipsoid. The magnetic susceptibility of an ideal diamagnet is $\chi = -1$. The velocity field of the liquid can be mapped to the magnetic field of the superconductor, since the field is curl- and divergence-free and tangential to the surface of the body. The kinetic energy of the water is then given by the energy of the magnetic field.

If we place the superconductor into a homogeneous magnetic field $\vec{B}_0(\vec{r})$, due to surface currents a field $\vec{B}_s(\vec{r})$ is created, which is homogeneous inside the superconductor and compensates the external field inside the superconductor. We need the kinetic energy of the fluid in the rest frame, where we subtract the velocity field relative to the body, which is identified with $\vec{B}_0(\vec{r})$.

We would need to exclude the energy of the magnetic field inside the superconductor, but since we take the limit of a thin disk, this contribution also vanishes.

Following the hint, we assume that a homogeneously polarized ellipsoid creates a homogeneous field inside itself. There are multiple ways to show this, for example to brute force calculate the volume integral over all infinitesimal magnetic dipole contributions. We take this as given here.

We obtain the field energy by

$$E_m = \frac{1}{2}\vec{m} \cdot \vec{B}_0 \tag{1}$$

which can be depicted by turning small magnets in the body, which needs an energy of $\vec{m}_0 \cdot \vec{B}(t)$ per magnet. The magnetization scales with the magnetic field and thus we get a factor 1/2 if we integrate from 0 to B_0. The magnetic moment of the ellipsoid is given by

$$\vec{m} = V_{\text{ellipsoid}}\vec{M} = \frac{4}{3}\pi R^2 h \vec{M} \tag{2}$$

where h is the height and in the limit of small h, the ellipsoid becomes a thin disk of radius R.

We have to calculate $\vec{B}_0$ for a given constant magnetization $\vec{M} = M\hat{e}_z$ of the ellipsoid. Since the magnetic field is homogeneous, we simply choose the center of the ellipsoid. Consider all points with distance r to the axis of the magnetization through the center of the ellipsoid. The surface density of the magnetization $J(r) = Md(r)$ can be evaluated by calculating the thickness d

$$\left(\frac{r}{R}\right)^2 + \left(\frac{z}{h}\right)^2 = 1 \quad \Leftrightarrow \quad d(r) = 2z = 2h\sqrt{1 - \left(\frac{r}{R}\right)^2} \tag{3}$$

Ampères law in the differential from gives us the magnetization current density, where we assume that now the thin disk has a magnetization $\vec{J} = J(r)\delta(z)\hat{e}_z$:

$$\vec{j}_m = \vec{\nabla} \times \vec{J} \tag{4}$$

The curl for a vector field $\vec{A} = f\left(\sqrt{x^2+y^2}\right)\delta(z)\hat{e}_z$ with only z-components is given by $\vec{\nabla} \times \vec{A} = \frac{\partial A_z}{\partial y}\hat{e}_x - \frac{\partial A_z}{\partial x}\hat{e}_y = \frac{\partial f(r)}{\partial r}(-\hat{e}_\phi)\delta(z)$. Then

$$\vec{j}_m(r) = -\frac{\partial}{\partial r}J(r)\hat{e}_\phi = \frac{2hM\delta(z)}{R}\frac{r/R}{\sqrt{1-(r/R)^2}}\hat{e}_\phi. \tag{5}$$

This result shows that we can think of the magnetization as realized by ring currents around the center of the disk. But we know that the magnetic field in the center of a ring current in the xy-plane with radius r is given by $\vec{B} = B\hat{e}_z$ with

$$B = \frac{\mu_0 I}{2r}. \tag{6}$$

If we now integrate over all contributing ring currents (so the current density), we obtain the result

$$B_0 = \int_0^R \frac{\mu_0 j_m(r)}{2r}\mathrm{d}r = \frac{\mu_0 M h}{R}\int_0^1 \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \overset{x=\sin\theta}{=} \frac{\mu_0 M h}{R}\int_0^{\pi/2}\mathrm{d}\theta = \frac{\mu_0 \pi M h}{2R} = \frac{3}{8}\frac{\mu_0 m}{R^3} \tag{7}$$

$$E_m = \frac{1}{2}\vec{m}\cdot\vec{B}_0 = \frac{1}{\mu_0}\frac{1}{2}\frac{8}{3}R^3 B_0^2 \tag{8}$$

For a velocity field we would obtain $E = \frac{1}{2}m_{\mathrm{add}}v_0^2$ we now have to replace the energy per unit volume and unit of the field $\frac{1}{2\mu_0}$ in case of the magnetic field by the corresponding term for a velocity field, namely the mass density $\frac{1}{2}\rho$, in our case the density of water. We arrive at the final expression for the added mass of a thin disk of diameter $D = 2R$ completely surrounded by water

$$m_{\mathrm{add}} = \frac{8}{3}\rho_W R^3 = \frac{1}{3}\rho_W D^3 \tag{9}$$

2 Extract measurements from the video

Using the open source software *Tracker* by Physlets, I measure the velocity before and after hitting the surface as well as the radius-to-height ratio of the cylinder. In figure 1 I show some screenshots of the program. Since the frame rate is high, we can neglect the gravity acceleration of the mass and do just a linear fit to the height values. That this fit works sufficiently is seen in the lower plots.

The following results were obtained, where the error is roughly estimated by the fit or the hand-measurement of the height and width.

$$v_0 = (8.36 \pm 0.08) \tag{10}$$
$$v_1 = (5.5 \pm 0.5) \tag{11}$$
$$H = (0.40 \pm 0.01) \tag{12}$$
$$D = (0.98 \pm 0.01) \tag{13}$$

As already stated in the hint, the measurement of v_1 is quite complicated, since it is based only on 3 frames. This might result in a significant error on the result.

3 Momentum conservation law

Finally, we apply the momentum conservation law. The momentum of the cylinder before it hits the water is $p_0 = m_c v_0$. Where $m_c = \rho_c H\pi\left(\frac{D}{2}\right)^2$ is the mass of the cylinder. After the lower side of the cylinder (a thin disk) hits the water, also the momentum of the water that is now in motion has to be considered. As explained in the hint, the impact is plastic and the excess energy is carried away by a shock wave in the water. The energy-to-momentum ratio is given by the speed of sound in water, which is much higher than the cylinder speed. Thus it can be neglected. Also, only the space below the cylinder is covered by water, the upper half space is covered by air. Since air is significantly lighter than water, we can neglect the contribution and have to use only half of the added mass we obtained before. Note that this is as explained in the other hints only an approximation for the short times after the cylinder hits the water. Almost immediately, the water surface is not perfectly flat anymore and the cylinder gets more and more immersed. However directly after the impact the momentum of the cylinder and the water, that was set into motion by the impact, can be calculated by $p_1 = \left(m_c + \frac{m_{\mathrm{add}}}{2}\right)v_1$.

Using that the momentum is conserved $p_0 = p_1$, we arrive at the following expression for the density of the cylinder

$$\rho_c = \frac{2}{3\pi}\frac{D}{H}\rho_W\left(\frac{v_0}{v_1} - 1\right)^{-1} \tag{14}$$

4 Result

Inserting the measured values we arrive at the final estimate of the density of the cylinder

$$\rho_c = (1.0 \pm 0.2)\rho_W = (1.0 \pm 0.2)\,\mathrm{g/mm^3} \tag{15}$$

The error is estimated by Gaussian error propagation and $\rho_W \approx 1\,\mathrm{g/mm^3}$. As explained before the error on the density estimate it very high due to the difficulty to measure v_1.

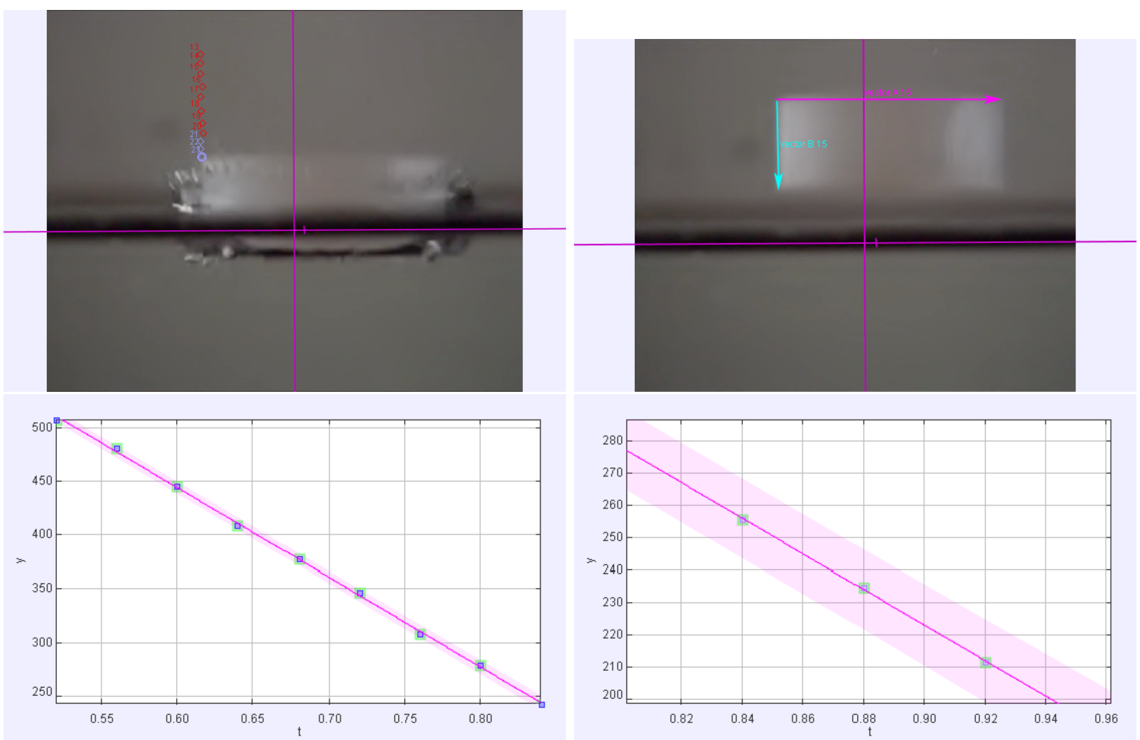

Figure 1: Screenshots of the *Tracker* software. (a) shows the tracked points. For this the upper left edge of the cylinder was used. (b) shows the height and width measurement. (c) shows the linear fit for the velocity before hitting the water and (d) shows the fit for the velocity after hitting the water.