Problem 3 – Scrap iron sorting

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Task

An assortment of scrap iron is scattered on a wooden table. The scrap consists of small pieces of thin sheet (with a thickness much smaller than their diameter) and tiny spherical balls. To remove the scrap, a spherical permanent magnet is used, approaching it from above with the magnetisation vector pointing downwards. The plates are lifted when the height of the centre of the magnet from the table surface is a. What is the height b at which the magnet lifts the spheres? It is assumed that the dimensions of the scrap particles are significantly smaller than a. The iron from which the scrap is made of is a soft ferromagnetic with a negligible width of the hysteresis loop; its relative permeability is much greater than one. The interaction between neighbouring scrap particles can be neglected.

Solution

Magnetic field

Let the table be at the height $z = 0$ and the scrap iron scattered in the neighborhood of the origin. The center of the spherical magnet is located at $h\boldsymbol{e}_z$, where h changes over time. Assume the magnet is uniformly magnetized having the downward pointing dipole moment $\boldsymbol{m} = -m\boldsymbol{e}_z$. Then, outside of the sphere, the magnetic field is given by $\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{f}(\boldsymbol{r} - h\boldsymbol{e}_z)$, where

$$\boldsymbol{f}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\boldsymbol{r}\langle\boldsymbol{m},\boldsymbol{r}\rangle}{r^5} - \frac{\boldsymbol{m}}{r^3}\right).$$

This is a pure dipole field. Even if the spherical magnet is not uniformly magnetized, the field approximates that of a dipole at large distances ($\gg$ radius of the magnet). Furthermore, we assume that the distance of the magnet from the table h is much greater than the extent of the scrap iron distribution, so that the magnetic field only depends on z in the neighborhood of the origin, where the pieces of iron are located:

$$\boldsymbol{B}(x, y, z) \approx \boldsymbol{B}(z\boldsymbol{e}_z) = -\frac{\mu_0 m\boldsymbol{e}_z}{2\pi\,(h - z)^3} \quad \text{for } z < h.$$

Magnetization of the scrap

Since the pieces of iron are so small, we assume that they are exposed to the constant magnetic field $\boldsymbol{B}_0 := \boldsymbol{B}(\boldsymbol{0})$ and are uniformly magnetized. In the following, we are looking for how the magnetization $\boldsymbol{M}$ of the balls and thin sheets depends on the external magnetic field $\boldsymbol{B}_0$. For this, suppose that the width of the hysteresis loop is negligible (i.e. magnetization vanishes at $\boldsymbol{H} = \boldsymbol{0}$) and that the induced magnetization remains much smaller than the saturation magnetization. Then a linear dependence between $\boldsymbol{M}$ and $\boldsymbol{H}$ can be assumed:

$$\boldsymbol{M} = (\mu_r - 1)\,\boldsymbol{H} \quad \text{with } \mu_r \gg 1.$$

Hence, $\boldsymbol{B} = \mu_0\,(\boldsymbol{M} + \boldsymbol{H}) = \mu_0\mu_r\boldsymbol{H}$.

Tiny balls The fields generated by the uniformly magnetized spheres are

$$\boldsymbol{H}_M = -\frac{\boldsymbol{M}}{3} \quad \text{and} \quad \boldsymbol{B}_M = \frac{2\mu_0 \boldsymbol{M}}{3}$$

inside them. Hence,

$$\begin{aligned}\boldsymbol{B} &= \boldsymbol{B}_0 + \boldsymbol{B}_M = \boldsymbol{B}_0 + \frac{2\mu_0 \boldsymbol{M}}{3} \quad \text{and} \\ \boldsymbol{H} &= \frac{\boldsymbol{B}_0}{\mu_0} + \boldsymbol{H}_M = \frac{\boldsymbol{B}_0}{\mu_0} - \frac{\boldsymbol{M}}{3}\end{aligned}$$

inside the balls. From $\boldsymbol{B} = \mu_0 \mu_r \boldsymbol{H}$ it follows that

$$\boldsymbol{M} = \frac{3(\mu_r - 1)}{\mu_0(\mu_r + 2)} \boldsymbol{B}_0 =: c_b \boldsymbol{B}_0.$$

Thin plates Here, $\boldsymbol{H}_M = -\boldsymbol{M}$ and $\boldsymbol{B}_M = \boldsymbol{0}$[1], i.e.

$$\boldsymbol{B}_0 = \mu_0 \mu_r \left(\frac{\boldsymbol{B}_0}{\mu_0} - \boldsymbol{M} \right)$$

inside the thin sheets. Hence,

$$\boldsymbol{M} = \frac{\mu_r - 1}{\mu_0 \mu_r} \boldsymbol{B}_0 =: c_a \boldsymbol{B}_0.$$

Forces on the scrap iron

The potential energy of a magnetic dipole with dipole moment $\boldsymbol{\mu}$ in a magnetic field $\boldsymbol{B}$ is given by

$$U = -\langle \boldsymbol{\mu}, \boldsymbol{B} \rangle.$$

Hence, the force acting on it is

$$\boldsymbol{F} = -\operatorname{grad}(U) = \operatorname{grad}\langle \boldsymbol{\mu}, \boldsymbol{B} \rangle.$$

Let V be the volume of an iron piece. Being uniformly magnetized, $\boldsymbol{\mu} = V\boldsymbol{M}$. The scrap particle experiences the upward pulling force

$$\boldsymbol{F} = c_{a/b} V B_0 \left. \frac{\partial B(z)}{\partial z} \right|_{z=0} \boldsymbol{e}_z,$$

depending on whether it is a thin plate (a) or a ball (b). Here,

$$B(z) = \frac{\mu_0 m}{2\pi \left(h - z\right)^3}$$

is the magnitude of $\boldsymbol{B}(z\boldsymbol{e}_z)$ and $B_0 = B(0)$ that of $\boldsymbol{B}_0$. Thus,

$$\boldsymbol{F} = \frac{3\mu_0^2 m^2}{4\pi^2} \cdot c_{a/b} V h^{-7} \boldsymbol{e}_z.$$

As the table is wooden and interactions between neighbouring scrap particles can be neglected, the only force that is left to be considered is gravity. Hence, the moment $|\boldsymbol{F}|$ becomes greater than $\rho V g$,

[1] This can be seen quickly. From $\operatorname{curl}(\boldsymbol{H}_M) = \boldsymbol{0}$ (no electric currents) and $\operatorname{curl}(\boldsymbol{M}) = \boldsymbol{0}$ (geometry of magnetization) it follows that $\operatorname{curl}(\boldsymbol{B}_M) = \boldsymbol{0}$. Together with $\operatorname{div}(\boldsymbol{B}_M) = 0$, this yields that $\boldsymbol{B}_M = \boldsymbol{0}$ everywhere (Helmholtz decomposition). Finding $\boldsymbol{H}_m$ and $\boldsymbol{B}_M$ inside and outside of an uniformly magnetized sphere is a well known problem in magnetostatics.

with ρ being the density of the iron, the scrap particle is lifted. Then the distance of the magnet from the table is

$$h = \left(\frac{3\mu_0^2 m^2}{4\pi^2 \rho g} \cdot c_{a/b} \right)^{1/7} .$$

As h does not depend on the volume, all thin plates are lifted in the same moment, when $h = a$, and all small balls when $h = b$. Hence,

$$\left(\frac{b}{a} \right)^7 = \frac{c_b}{c_a} = \frac{3(\mu_r - 1)}{\mu_0(\mu_r + 2)} \cdot \frac{\mu_0 \mu_r}{\mu_r - 1} = \frac{3\mu_r}{\mu_r + 2} \approx 3$$

as $\mu_r \gg 1$. The magnet therefore lifts the tiny spheres first when it reaches height $b = 3^{1/7} a \approx 1.17a$.