Physics Cup 2024 - Problem 3

Kutay Kulbak

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The scraps will get magnetized by the magnetic field of the permanent magnet. The interaction force between the permanent magnet and the magnetized scraps will pull the scraps upwards, and the scraps will get lifted when the magnetic force becomes equal to the weight of the scraps.

The magnetic field of the permanent magnet will be equal to the field of a magnetic dipole:

$$\mathbf{B}_m(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m_0}{r^3}[3(\hat{m}_0 \cdot \hat{r})\hat{r} - \hat{m}_0]$$

Here, $m_0 = |\mathbf{m}_0|$ is the magnitude of the magnetic dipole vector of the magnet, $\hat{m}_0$ is the unit vector of the magnetic dipole vector pointing towards the scraps.

Since the magnetic field is stronger when r is smaller, the scraps directly below the magnet will be lifted first. For those scraps, we can simplify the magnetic field as below:

$$\mathbf{B}_m(r) = \frac{\mu_0}{2\pi}\frac{\mathbf{m}_0}{r^3}$$

1. Plates: From the boundary condition

$$\mathbf{B}_{in}^{\perp} = \mathbf{B}_{out}^{\perp}$$

we know that the magnetic field inside the thin plates will be equal to

$$\mathbf{B} = \mathbf{B}_m$$

Also, the fact that the plates are linear ferromagnets with $\mu \gg 1$ gives us

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu\mu_0\mathbf{H}$$
$$\Rightarrow \mathbf{M} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{\mu_0} \approx \frac{\mathbf{B}}{\mu_0}$$

The fact that the dimensions of the sheet scraps are much less than a allows us to model them as perfect magnetic dipoles with $\mathbf{m} = V\mathbf{M}$, where V is the volume of a scrap. Hence, the interaction force between the permanent magnet and a scrap can be shown as

$$\begin{aligned}\mathbf{F}(r = a) &= \mathbf{m}(\nabla \cdot \mathbf{B}_m)\\ &= V \cdot \frac{1}{\mu_0}\left(\frac{\mu_0}{2\pi}\frac{\mathbf{m}_0}{a^3}\right) \cdot \frac{\partial}{\partial r}\left(\frac{\mu_0}{2\pi}\frac{m_0}{r^3}\right)\bigg|_a\\ &= \frac{V\mathbf{m}_0}{2\pi a^3} \cdot \frac{-3\mu_0 m_0}{2\pi a^4}\end{aligned}$$

$$\Rightarrow \mathbf{F} = -\frac{3\mu_0 V m_0^2}{4\pi^2 a^7}\hat{m}_0$$

The minus sign at the beginning shows that the force is an attracting force, pulling the scrap up. Setting this force equal to the weight of the scrap, we get

$$\frac{3\mu_0 m_0^2}{4\pi^2 a^7} = \rho g$$

2. Spheres: According to Maxwell's laws, we know that

$$\begin{cases} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{j}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

We know that there is no free current, no electric field, and also that $\mathbf{B} = \mu\mu_0\mathbf{H}$. From these, we get

$$\begin{cases} \nabla \cdot \mathbf{H} &= 0 \\ \nabla \times \mathbf{H} &= 0 \end{cases}$$

These equations allow us to look for a solution of the equation

$$\nabla^2\psi = 0$$

where $\mathbf{H} = -\nabla\psi$, and ψ is the magnetic scalar potential.

We also have the following boundary conditions:

$$\begin{cases} \mathbf{H}_{out}^{\perp} = \mu \mathbf{H}_{in}^{\perp} \\ \mathbf{H}_{out}^{\parallel} = \mathbf{H}_{in}^{\parallel} \\ \mathbf{H}_{out}(r \to \infty) = \mathbf{H}_m \end{cases}$$

The solution that fits these boundary conditions is

$$\begin{cases} \psi_{out} &= -\mathbf{H}_m \cdot \mathbf{r} + A\dfrac{\mathbf{H}_m \cdot \mathbf{r}}{r^3} \\ \psi_{in} &= B\mathbf{H}_m \cdot \mathbf{r} \end{cases}$$

Here, the first part of ψ_{out} is the limit value at $r \to \infty$, and the second part is the only solution that satisfies the Laplace equation, depends only on one constant vector ($\mathbf{H}$), and vanishes at infinity. ψ_{in} is the only solution which satisfies the Laplace equation and is finite at the center of the sphere. Plugging these potentials back to the equation,

$$\begin{cases} \mathbf{H}_{out} &= -H_m \left[\left(\dfrac{2A}{r^3} + 1 \right) \cos\theta \hat{r} + \left(\dfrac{A}{r^3} - 1 \right) \sin\theta \hat{\theta} \right] \\ \mathbf{H}_{in} &= H_m \left(\dfrac{A}{r^3} - 1 \right) (\cos\theta \hat{r} - \sin\theta \hat{\theta}) \end{cases}$$

From the boundary conditions, we get that

$$\begin{cases} B = \dfrac{A}{R^3} - 1 \\ A = \dfrac{\mu - 1}{\mu + 2} R^3 \end{cases}$$

Here, R is the radius of the sphere. These give us the field inside the sphere as

$$\mathbf{H}_{in} = \frac{3}{\mu + 2}\mathbf{H}_m$$

From this, we find the magnetization of the sphere as

$$\mathbf{M} = \frac{\mu - 1}{\mu + 2}\frac{3\mathbf{B}_m}{\mu_0}$$

Using the expression for the force we found in the previous part, we get

$$\begin{aligned}\mathbf{F} &= V \cdot \frac{\mu - 1}{\mu + 2}\frac{3}{\mu_0} \cdot \frac{\mu_0}{2\pi}\frac{\mathbf{m}_0}{b^3} \cdot \frac{-3\mu_0 m_0}{2\pi b^4} \\ &= -\frac{9\mu_0 V m_0^2}{4\pi^2 b^7}\frac{\mu - 1}{\mu + 2}\hat{m}_0\end{aligned}$$

This gives us

$$\frac{9\mu_0 m_0^2}{4\pi^2 b^7}\frac{\mu - 1}{\mu + 2} = \rho g = \frac{3\mu_0 m_0^2}{4\pi^2 a^7}$$

Finally, using that $\mu \gg 1$, we find

$$\boxed{b = 3^{1/7}a \approx 1.17a}$$