Physics Cup — TalTech 2024 — Problem 3

Solution by Konstantin Rodionenko, ITMO University, Saint-Petersburg

> An assortment of scrap iron is scattered on a wooden table. The scrap consists of small pieces of thin sheet (with a thickness much smaller than their diameter) and tiny spherical balls. To remove the scrap, a spherical permanent magnet is used, approaching it from above with the magnetisation vector pointing downwards. The plates are lifted when the height of the centre of the magnet from the table surface is a. What is the height b at which the magnet lifts the spheres? It is assumed that the dimensions of the scrap particles are significantly smaller than a. The iron from which the scrap is made of is a soft ferromagnetic with a negligible width of the hysteresis loop; its relative permeability is much greater than one. The interaction between neighbouring scrap particles can be neglected.

I work in Gaussian units. Consider the tiny spherical iron ball (μ — iron magnetic permeability) in homogeneous magnetic field $\boldsymbol{B_0}$. The ball magnetization turns out to be uniform. Moreover, magnetic field $\boldsymbol{B'}$ formed by the ball inside connected to the magnetization $\boldsymbol{M}$:

$$\boldsymbol{B'} = \frac{8\pi}{3}\boldsymbol{M}. \tag{1}$$

The above can be proven by constructing the analogy to the dielectric ball in the electric field. In this case, the magnetization corresponds to the polarisation. Let's notice that the homogeneous electric field inside the ball can be created by two uniformly charged balls which are offset from each other. Denote $\boldsymbol{r_1}$, $\boldsymbol{r_2}$ — vectors from balls centers to the considered point inside the balls. We get homogeneous electric field created by ball.

$$\boldsymbol{E'} = \frac{4\pi\rho}{3}\boldsymbol{r_1} - \frac{4\pi\rho}{3}\boldsymbol{r_2} = \frac{4\pi\rho}{3}\boldsymbol{\Delta r} = -\frac{4\pi}{3}\boldsymbol{P}. \tag{2}$$

In the case of magnetic ball we should get the field inside ball by using the formula for magnetic field created by magnetic dipole in the pole of ball (normal component of $\boldsymbol{B}$ is continuous).

$$\boldsymbol{B'} = \frac{3\boldsymbol{r}(\boldsymbol{m}\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3} = \frac{2\boldsymbol{m}}{r^3} = \frac{8\pi}{3}\boldsymbol{M} \quad (\boldsymbol{m} \parallel \boldsymbol{r}). \tag{3}$$

Let's take into account that $\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M} = \mu\boldsymbol{H}$ and $\boldsymbol{H} = \boldsymbol{B_0}$. We instantly derive

$$\boldsymbol{M} = \frac{1}{4\pi} \cdot \frac{3(\mu-1)}{\mu+2}\boldsymbol{B_0}. \tag{4}$$

Now consider a small piece of iron thin sheet. Because a thickness much smaller than their diameter we neglect edge effects and get

$$\boldsymbol{M} = \frac{1}{4\pi} \cdot \frac{\mu-1}{\mu}\boldsymbol{B_0}. \tag{5}$$

In our task assortment of scrap iron magnetized by the spherical permanent magnet. Because the dimensions of the scrap particles are significantly smaller that the distance r to the magnet, the force emerging between every particle and the magnet can be calculated as the force between two magnetic dipoles. Apart from this, we assume the field near every iron particle to be homogeneous. Therefore[1]

$$\boldsymbol{F} = \nabla\left(\boldsymbol{m}\boldsymbol{B}\right), \tag{6}$$

where $\boldsymbol{m} = \boldsymbol{M}V$ — the particle dipole moment, $\boldsymbol{M}$, V — its magnetization vector and volume, respectively; $\boldsymbol{B}$ — magnetic field generated by the magnet around the particle, it is proportional to r^{-3}.

From the first part of the solution we understand that $\boldsymbol{M} \sim \boldsymbol{B} \sim r^{-3}$. Thus

$$\frac{\boldsymbol{F}}{V} = \nabla\left(\boldsymbol{M}\boldsymbol{B}\right) \sim r^{-7}. \tag{7}$$

Proportionality coefficients ratio in two cases (iron ball and piece of thin sheet) is equal to $\frac{3(\mu-1)}{\mu+2} / \frac{\mu-1}{\mu}$. In the limit $\mu \gg 1$ this fraction strives for 3.

The gravitational force is uniform for both the particle and the ball. Using an equilibrium condition, we get

$$\frac{1}{a^7} = \frac{3}{b^7} \Rightarrow b = a\sqrt[7]{3}. \tag{8}$$

Answer. $b = a\sqrt[7]{3}$.

[1] It is important to mention that nabla differential operator acts only to $\boldsymbol{B}$, because magnetic moment m was measured in the case of homogeneous field. At the same time, this remark does not affect the answer.