Physics Cup 2024

Problem 4

Satellites

Ayaan Maan

March 2, 2024

1 Problem

Two satellites orbit Earth on the same plane along elliptic paths of eccentricities e_1 and e_2 respectively. The angle between their major axes is α. The rate at which a line segment connecting a satellite and the Earth's centre sweeps out an area is L_1 and L_2 respectively. What is the maximal relative velocity of the satellites? Provide also a simplified answer for $\alpha = \frac{\pi}{2}$ and $L_1 = L_2$. The ratio of the orbital periods of the satellites is an irrational number. Earth's mass is denoted by M, and the gravitational constant by G.

2 Brief theory and assumptions

We make the trivial assumption that $M >> m \; \forall \; m \in \{m_1, m_2\}$ where m_1 and m_2 are the masses of the two satellites. For our system, the earth is taken to be the origin and expressions for angular momentum are as calculated about the earth unless otherwise noted.

As a standard result in coordinate geometry, we have the polar equation of a conic as

$$r = \frac{a(1-e^2)}{1 - e\cos(\theta)} \tag{1}$$

For a particular satellite of mass m, the angular momentum is

$$\mathcal{L} = m\sqrt{GMa(1-e^2)} \tag{2}$$

and areal velocity

$$L = \frac{\mathcal{L}}{2m} \tag{3}$$

Using conservation of angular momentum,

$$\mathcal{L} = mv_\theta r \tag{4}$$

and conservation of energy

$$\mathcal{E} = \frac{1}{2}m(v_r^2 + v_\theta^2) - \frac{GMm}{r} = \frac{-GMm}{2a} = \frac{1}{2}\langle U \rangle \tag{5}$$

we get

$$\boxed{v_\theta = c(1 - e\cos(\theta))} \tag{6}$$

$$\boxed{v_r = ce\sin(\theta)} \tag{7}$$

where

$$\boxed{c = \frac{GM}{2L}} \tag{8}$$

3 Geometry of problem

3.1 Directions of revolution

There can be two cases corresponding to the satellites motion-both of them travel anticlockwise (or clockwise, the calculations remain same due to symmetry) or one anticlockwise and the other clockwise. In the first case our analysis proceeds as normal but in the second case, we have to substitute $\pi - \alpha$ in our final answer. This can be seen by the following construction of reflecting the orbit of satellite 1 about y-axis and viewing the setup again 'from the other side of the page'. Reflection preserves magnitude and the 'directionality' of the setup is now identical to that of the first case. The new angle between the major axis of orbits becomes $\pi - \alpha$.

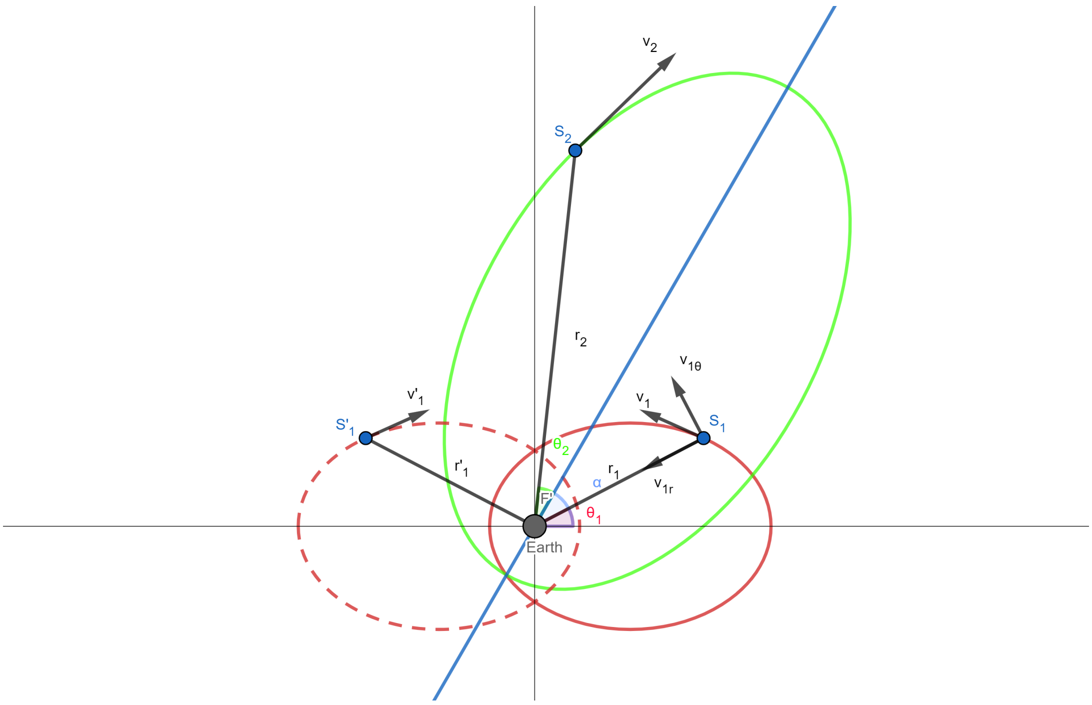

Figure 1: NB: Velocity vectors not to scale

3.2 Coordinate analysis

For further analysis, we make use of cartesian coordinates. We write the velocities of each satellite in x-y components where the x axis is the major axis of the first satellite's orbit. We get the following expressions:

$$\overrightarrow{v_1} = \begin{pmatrix} c_1 \sin(\theta_1) \\ c_1 (\cos(\theta_1) - e_1) \end{pmatrix} \tag{9}$$

$$\overrightarrow{v_2} = \begin{pmatrix} c_2 (e_2 \sin(\alpha) - \sin(\theta_2 + \alpha)) \\ c_2 (\cos(\theta_2 + \alpha) - e_2 \cos(\alpha)) \end{pmatrix} \tag{10}$$

Hence, $\overrightarrow{v_{rel}} = \overrightarrow{v_1} - \overrightarrow{v_2}$. To simplify expresions, we can substitute $V = v_{rel}^2$ in the derivatives. To get maximum value of V, we need to solve the system of equations

$$\frac{\partial V}{\partial \theta_1} = 0 \tag{11}$$

$$\frac{\partial V}{\partial \theta_2} = 0 \tag{12}$$

or equivalently,

$$c_1 e_1 \sin(\theta_1) - c_2 e_2 \sin(\theta_1 + \alpha) + c_2 \sin(\theta_1 + \theta_2 + \alpha) = 0 \tag{13}$$

$$c_2 e_2 \sin(\theta_2) - c_1 e_1 \sin(\theta_2 + \alpha) + c_1 \sin(\theta_1 + \theta_2 + \alpha) = 0 \tag{14}$$

4 Analysing equations for maximae

Notice that by subtracting equations (11) and (12), we can factor out a term $\sin(\frac{\theta_1+\theta_2+\alpha}{2})$ using the half angle identities and obtain the following equation

$$2\sin\left(\frac{\theta_1+\theta_2+\alpha}{2}\right)((c_2-c_1)\cos\left(\frac{\theta_1+\theta_2+\alpha}{2}\right)-$$
$$c_2e_2\cos\left(\frac{\theta_1-\theta_2+\alpha}{2}\right)+c_1e_1\cos\left(\frac{-\theta_1+\theta_2+\alpha}{2}\right))=0$$

Hence, we get a solution when $\theta_1+\theta_2+\alpha$ is an even multiple of π. Replacing this identity in equations (11) and (12) yields the general solutions

$$\Theta_1=\cot^{-1}\left(\frac{c_1e_1-c_2e_2\cos(\alpha)}{c_2e_2\sin(\alpha)}\right)+k_1\pi \tag{15}$$

$$\Theta_2=\cot^{-1}\left(\frac{c_2e_2-c_1e_1\cos(\alpha)}{c_1e_1\sin(\alpha)}\right)+k_2\pi \tag{16}$$

Since range of function $f:\mathbb{R}\to\mathbb{R}, f(x)=\cot^{-1}(x)$ is $[0,\pi]$ and physically $\theta_1,\theta_2\in[0,2\pi]$, then $k_1,k_2\in\{0,1\}$.

Notice that the general solution implies $\alpha+\Theta_1+\Theta_2=(k_1+k_2+1)\pi$. The case $k_1+k_2=2\gamma+1$ is satisfied by the first bracket while $k_1+k_2=2\gamma$ satisfies the second one.

4.1 Second partial derivative test

We need to perform the second partial derivative test for two-variable functions to check different cases of values of k_1 and k_2. The conditions to be satisfied are

$$V(\theta_1,\theta_2)=|\overrightarrow{v_1}-\overrightarrow{v_2}|^2 \tag{17}$$

$$D_{\mathbf{H}}=\det\left(\mathbf{H}_V(\Theta_1,\Theta_2)\right)>0 \tag{18}$$

$$\mathbf{d}=\left(\mathbf{H}_V(\Theta_1,\Theta_2)\right)_{1,1}<0 \tag{19}$$

where $\mathbf{H}_V(\theta_1,\theta_2)$ is the Hessian matrix of V.

4.2 Different cases of k_1 and k_2

1. $k_1 + k_2 = 2\gamma + 1$
   (a) $D_{\mathbf{H}} > 0$
   (b) $\mathbf{d} > 0$
2. $k_1 + k_2 = 2\gamma$
   (a) $k_1 = k_2 = 0$
      i. $D_{\mathbf{H}} < 0$
   (b) $k_1 = k_2 = 1$
      i. $D_{\mathbf{H}} > 0$
      ii. $\mathbf{d} < 0$

Therefore, our required solution for Θ_1 and Θ_2 are

$$\boxed{\Theta_1 = \cot^{-1}\left(\frac{c_1e_1 - c_2e_2\cos(\alpha)}{c_2e_2\sin(\alpha)}\right) + \pi} \tag{20}$$

$$\boxed{\Theta_2 = \cot^{-1}\left(\frac{c_2e_2 - c_1e_1\cos(\alpha)}{c_1e_1\sin(\alpha)}\right) + \pi} \tag{21}$$

5 Maximum relative velocity

Substituting the obtained expressions for Θ_1 and Θ_2 in (17), we get

1. Both clockwise or anticlockwise

$$\boxed{v_{rel,max} = c_1 + c_2 + \sqrt{c_1^2 e_1^2 + c_2^2 e_2^2 - 2c_1 c_2 e_1 e_2 \cos(\alpha)}} \tag{22}$$

2. One clockwise, one anticlockwise

$$\boxed{v_{rel,max} = c_1 + c_2 + \sqrt{c_1^2 e_1^2 + c_2^2 e_2^2 + 2c_1 c_2 e_1 e_2 \cos(\alpha)}} \tag{23}$$

3. **Specific case: $\alpha = \frac{\pi}{2}$, $L_1 = L_2 = L$:** We can replace these values in either (22) or (23) and get same answer because in our construction, the angle between axis remains same, i.e. $\frac{\pi}{2}$.

$$\boxed{v_{rel,max} = c\left(2 + \sqrt{e_1^2 + e_2^2}\right)} \tag{24}$$