Physics Cup Problem 4

Sainavaneet Mukund

26 February 2024

Contents

1 Invariance of the eccentricity vector

Consider the vector:

$$\mathbf{F} \times \mathbf{L} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{p} \times \mathbf{L}\right)$$

Expanding L.H.S:

$$\frac{-Gm_1m_2}{r^2}\hat{\mathbf{r}} \times (\mathbf{r} \times \mu\mathbf{v})$$

Using the: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ identity and expanding the position and velocity in terms of polar coordinates:

$$Gm_1m_2\dot{\theta}\hat{\theta} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{p} \times \mathbf{L}\right)$$

Using the fact that $\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} = \dot{\theta}\hat{\theta}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{p} \times \mathbf{L} - \mu Gm_1m_2\hat{\mathbf{r}}\right) = 0$$

which represents a conserved quantity,

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu Gm_1m_2\hat{\mathbf{r}}$$

Dividing $\vec{A}$ by μGm_1m_2 we get the eccentricity vector $\vec{\varepsilon}$ i.e

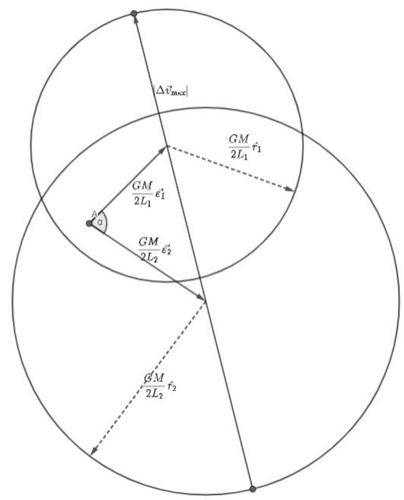
$$\boxed{\vec{\varepsilon} = \frac{\vec{v} \times \vec{h}}{GM} - \hat{r}}$$

Where $\vec{h}$ is the specific angular momentum which is related to the rate of change of area by Kepler's Second Law we which states that the rate of change of area is constant, thus:

$$L = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2}$$

2 Solution

Now, as the position of the satellites vary across the orbit, the locus of the tip of $\hat{r}$ is a circle so, let us consider the vector: $\vec{\varepsilon} + \hat{r}$ as in the diagram:



We may say that:

$$\vec{v} \times \hat{n} = \frac{GM}{2L}\left(\vec{\varepsilon} + \hat{r}\right)$$

Where $\hat{n}$ is the unit vector perpendicular to the plane of the orbit of both satellites.
Thus we may maximise $|\vec{v_2} - \vec{v_1}|$ by maximizing $|\vec{v_2} \times \hat{n} - \vec{v_1} \times \hat{n}|$. This is easily maximised by the line joining the centres of both circles.
Thus from the diagram it follows that:

$$\boxed{|\Delta\vec{v}|_{\max} = \sqrt{\left(\frac{GMe_2}{2L_2}\right)^2 + \left(\frac{GMe_1}{2L_1}\right)^2 - 2\frac{GMe_1}{2L_1}\frac{GMe_2}{2L_2}\cos\alpha} + \frac{GM}{2L_1} + \frac{GM}{2L_2}}$$

For the specific case of: $L_1 = L_2$ and $\alpha = \frac{\pi}{2}$:

$$\boxed{\Delta v_{\max} = \left(\sqrt{e_2^2 + e_1^2} + 2\right)\frac{GM}{2L}}$$