Physics cup 2024 Problem 4

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1 Introduction

The solution is organized as follows: First I introduce the Runge-Lenz vector (eccentricity vector), a very useful and not so well known concept in astrophysics problems such as this one. I will also prove some lemmas considering the Runge-Lenz vector. I will also relate the angular momentum and area sweeping rate L of the satellites. Then I will use these Lemmas to obtain an (achievable) upper bound on $|\vec{v}_2 - \vec{v}_1|$, the relative velocity of the satellites.

2 Runge-Lenz vector lemmas

Let us consider a body of mass m in an elliptical orbit around a much heavier object M (for example a satellite orbiting Earth). Then this heavier object is approximately motionless and is situated at one of the foci of the elliptical orbit. Let $\vec{r}$ be the vector connecting M and m. Let $\vec{v}$ be the velocity of m relative to M. Also let $\vec{J}$ be the angular momentum of m relative to M. The Runge-Lenz vector is defined as:

$$\vec{\varepsilon} = \frac{1}{GmM}\vec{v} \times \vec{J} - \hat{r} \tag{1}$$

Here $\hat{r} = \frac{\vec{r}}{r}$. Note that $\vec{J} = J\hat{z}$ (we are working in the cylindrical coordinate system (r,ϕ,z) with the origin at M).

2.1 Lemma 1

We claim that $\dot{\vec{\varepsilon}} = 0$ i.e. the conservation of the Runge-Lenz vector. Proof:

$$\begin{aligned}\dot{\vec{\varepsilon}} &= \frac{1}{GmM}\dot{\vec{v}} \times \vec{J} - \dot{\hat{r}} = \frac{1}{GMm}\left(-\frac{GM}{r^2}\right)(\hat{r} \times \hat{z})J - \dot{\phi}\hat{\phi} \\ &= \frac{J}{mr^2}\hat{\phi} - \dot{\phi}\hat{\phi} = 0\end{aligned}$$

Here we used $\dot{\vec{J}} = 0$ (conservation of angular momentum), Newtons second law for m, and some simple vector identities in cylindrical coordinates.

2.2 Lemma 2

We claim that $\vec{\varepsilon} \cdot \vec{\varepsilon} = e^2$, where e is the eccentricity of the elliptical orbit. Proof:

$$\vec{\varepsilon} \cdot \vec{\varepsilon} = \left(\frac{1}{GmM}\vec{v} \times \vec{J} - \hat{r}\right)^2 = \frac{v^2 J^2}{(GmM)^2} - \frac{2Jr\dot{\phi}}{GmM} + 1 = 1 + \frac{2EJ^2}{m(GmM)^2} = e^2$$

The last equality is a well known formula for the eccentricity (E is the energy of m).

2.3 Lemma 3

We claim $J = 2mL$. Proof: consider the triangle $\vec{r}(t), \vec{r}(t + \mathrm{d}t), \mathrm{d}\vec{r}(t)$. Its area is just $L\mathrm{d}t$. It is also just $\frac{1}{2}r^2\dot{\phi}\mathrm{d}t$. Thus $L = \frac{1}{2}r^2\dot{\phi} = \frac{J}{2m}$. Corollary:

$$\vec{\varepsilon} = \frac{2L}{GM}\vec{v} \times \hat{z} - \hat{r} \Rightarrow \vec{v} = \hat{z} \times (\vec{v} \times \hat{z}) = \frac{GM}{2L}\hat{z} \times (\vec{\varepsilon} + \hat{r}) = \frac{GM}{2L}(\vec{\chi} + \hat{\phi}) \tag{2}$$

Where $\vec{\chi}$ is the vector with magnitude e perpendicular to $\vec{\varepsilon}$.

3 Solution

In each moment by lemma 3:

$$\vec{v}_1 = \frac{GM}{2L_1}(\vec{\chi}_1 + \hat{\phi}_1), \ \vec{v}_2 = \frac{GM}{2L_2}(\vec{\chi}_2 + \hat{\phi}_2)$$

Then:

$$(\vec{v}_2 - \vec{v}_1)^2 = \left(\frac{GM}{2}\right)^2 \left[\underbrace{\left(\frac{\vec{\chi}_2}{L_2} - \frac{\vec{\chi}_1}{L_1}\right)}_{\vec{A}} + \underbrace{\left(\frac{\hat{\phi}_2}{L_2} - \frac{\hat{\phi}_1}{L_1}\right)}_{\vec{B}} \right]^2$$

We need only to bound $(\vec{A} + \vec{B})^2 = A^2 + B^2 + 2\vec{A} \cdot \vec{B}$. Since $\vec{A}$ is a fixed vector, and vector $\vec{B}$ is of bounded magnitude:

$$|\vec{B}| = \left| \frac{\hat{\phi}_2}{L_2} - \frac{\hat{\phi}_1}{L_1} \right| \leq \frac{1}{L_1} + \frac{1}{L_2}$$

Here we use that the period is irrational and thus the unit vectors $\hat{\phi}_1, \hat{\phi}_2$ achieve all possible configurations. Also $\vec{A} \cdot \vec{B} \leq AB$. Thus:

$$(\vec{A} + \vec{B})^2 \leq A^2 + 2A\left(\frac{1}{L_1} + \frac{1}{L_2}\right) + \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^2 = \left(A + \frac{1}{L_1} + \frac{1}{L_2}\right)^2$$

We can easily calculate:

$$A^2 = \left(\frac{\vec{\chi}_2}{L_2} - \frac{\vec{\chi}_1}{L_1}\right)^2 = \frac{e_1^2}{L_1^2} + \frac{e_2^2}{L_2^2} - 2\frac{e_1}{L_1}\frac{e_2}{L_2}\cos\alpha$$

Since α is the angle between $\vec{\chi}_1, \vec{\chi}_2$. Thus we may put this into the expression for the relative velocity:

$$|\vec{v}_2 - \vec{v}_1| \leq \frac{GM}{2L_1L_2}\left[\sqrt{e_1^2L_2^2 + e_2^2L_1^2 - 2e_1e_2L_1L_2\cos\alpha} + L_1 + L_2\right] \tag{3}$$

Thus since this value is in fact obtainable (for $\hat{\phi}_1, \hat{\phi}_2$ parallel to $\vec{A}$), it is the maximal possible value. More specifically, for $L_1 = L_2 = L$ and $\alpha = 90°$ (i.e. $\cos\alpha = 0$) we have:

$$|\vec{v}_2 - \vec{v}_1|_{max} = \frac{GM}{2L}\left[\sqrt{e_1^2 + e_2^2} + 2\right] \tag{4}$$

Note: This result can be easily geometrically interpreted by a vector diagram. The vector $\vec{B}$ lies in the circle of radius $\frac{1}{L_1} + \frac{1}{L_2}$ (Figure 1).

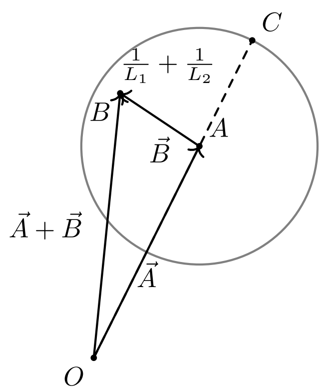

Figure 1: the vector diagram with $\vec{A}, \vec{B}$, note that B must lie inside the circle with centre A and radius $\frac{1}{L_1} + \frac{1}{L_2}$, obviously point C is the farthest possible point that the vector sum $\vec{A} + \vec{B}$ can achieve, its distance from the origin being $A + \frac{1}{L_1} + \frac{1}{L_2}$