Problem 4 – Satellites

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Task

Two satellites orbit Earth on the same plane along elliptic paths of eccentricities e_1 and e_2 respectively. The angle between their major axes is α. The rate at which a line segment connecting a satellite and the Earth's centre sweeps out an area is L_1 and L_2 respectively. What is the maximal relative velocity of the satellites? Provide also a simplified answer for $\alpha = 90°$ and $L_1 = L_2$. The ratio of the orbital periods of the satellites is an irrational number. Earth's mass is denoted by M, and the gravitational constant by G.

Solution

Assumptions

We assume that the satellites have a negligible mass compared to that of the Earth. Hence, we can exclude gravitational interactions between them and assume that the Earth is at rest in the center of mass frame.

Furthermore, since the ratio of the orbital periods of the satellites is an irrational number, the satellites can both be at arbitrary positions on their orbit at a certain point in time. This is because the satellites are never both at the same position twice in time. If this were the case, then the time $nT_1 = mT_2$ would have elapsed between these events, with T_1 and T_2 denoting the periods of the satellites and $n, m \in \mathbb{N}$, and T_1/T_2 would be rational. Thus, after the times nT_1, $n \in \mathbb{N}$, satellite 2 is always at a different point on its orbit and gets arbitrarily close to any position as time passes.

General aspects of the movement of a satellite

Before starting to explicitly deal with the problem, some basic properties of the motion of a satellite are outlined.

Let $\boldsymbol{r} = (r\cos\varphi, r\sin\varphi)$ be the time-varying position vector pointing from the center of the Earth to the satellite. The velocity of the latter relative to the Earth is $\boldsymbol{v} = \dot{\boldsymbol{r}}$. Having the mass m, the energy E and the angular momentum l, the satellite moves along the following elliptical orbit:

$$r(\varphi) = \frac{p}{1 + e\cos\varphi}, \text{ where } \; p = \frac{l^2}{Gm^2M} \; \text{ and } \; e = \sqrt{1 + \frac{2El^2}{G^2m^3M^2}} < 1.$$

E and l are constant over time as gravity is a conservative central force. At $\varphi = 0$ the satellite is in the perihelion of its orbit, i.e. r gets minimal. At $\varphi = \pi$ it is in the aphelion. e is called the eccentricity of the ellipse. Its semi-major axis is given by

$$a = \frac{r(0) + r(\pi)}{2} = \frac{1}{2}\left(\frac{p}{1+e} + \frac{p}{1-e}\right) = \frac{p}{1-e^2}.$$

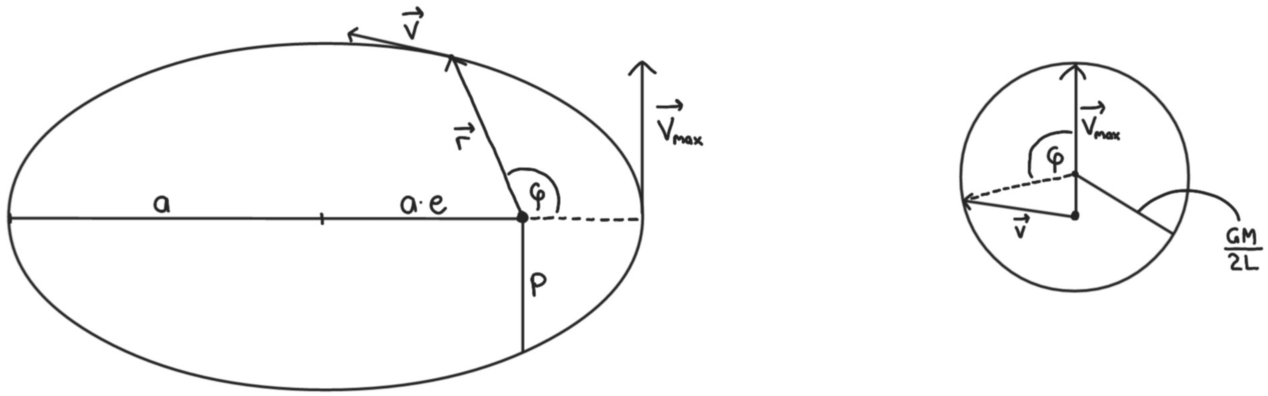

Figure 1: Elliptic path of a satellite and its velocities

This yields the following expressions for the angular momentum and the energy:

$$l = \sqrt{Gm^2Ma\,(1-e^2)} \quad \text{and} \quad E = -\frac{G^2m^3M^2\left(1-e^2\right)}{2l^2} = -\frac{GmM}{2a}. \tag{1}$$

We can also find the velocity of a satellite as a function of its distance from the Earth. As the total energy is given by

$$E = \frac{mv^2}{2} - \frac{GmM}{r} = -\frac{GmM}{2a},$$

we obtain by rearranging

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

This is the Vis-viva equation. The velocity of a satellite becomes maximal when it passes through the perihelion of its orbit. Then $r = a\,(1-e)$ and hence

$$v_{\max}^2 = \frac{GM}{a} \cdot \frac{1+e}{1-e}. \tag{2}$$

Velocity vectors of the satellites

In order to find the maximum relative velocity, our goal is to first display all the velocity vectors that the satellites assume in the course of a revolution around the Earth.

The movement of a satellite is determined by Newton's law of gravity. If it passes through the point $\boldsymbol{r} = (r\cos\varphi, r\sin\varphi)$, the change in velocity is given by

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{GM}{r^2} \cdot \begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix}. \tag{3}$$

In an infinitesimal time step, the line segment connecting the satellite and the Earth's centre sweeps out the area

$$\mathrm{d}A = \frac{|\boldsymbol{r} \times \boldsymbol{v}\mathrm{d}t|}{2}.$$

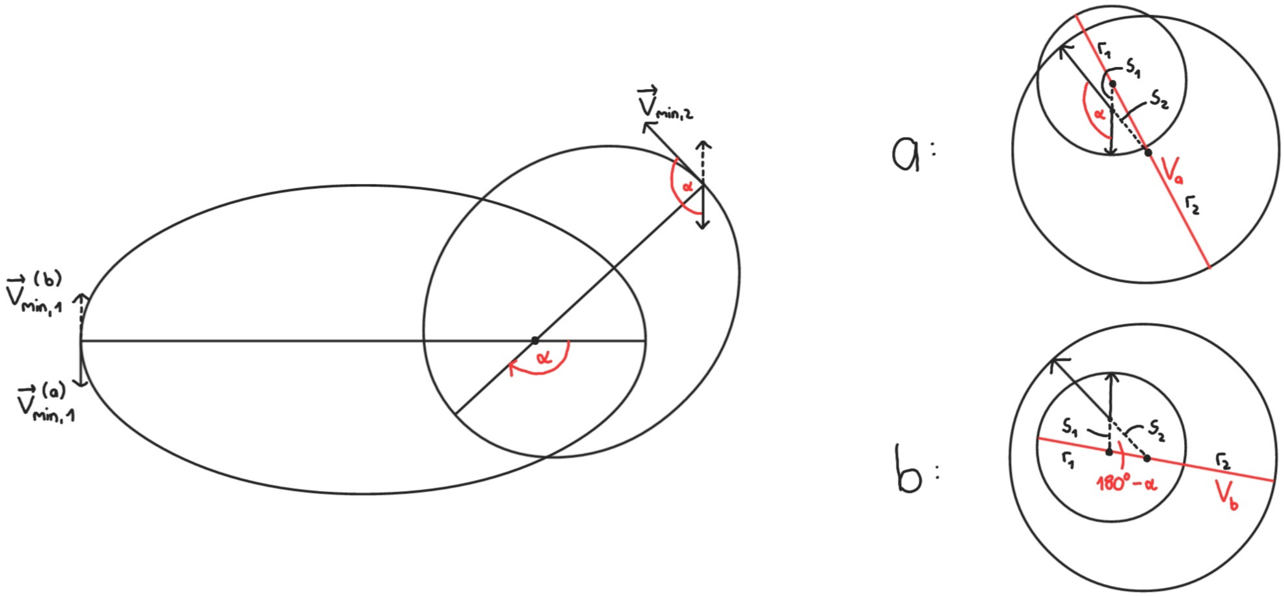

Figure 2: Finding the maximal relative velocity in the case that the two satellites orbit the Earth in the same direction (a) or in opposite directions (b).

As the angular momentum is given by $l = m|\boldsymbol{r} \times \boldsymbol{v}| = mr^2\dot{\varphi}$,

$$L := \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{l}{2m} = \frac{r^2\dot{\varphi}}{2},$$

which is constant over time. Applying the chain rule, we thus also obtain

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \dot{\varphi} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\varphi} = \frac{2L}{r^2} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\varphi}. \tag{4}$$

Comparison between (3) and (4) gives

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\varphi} = -\frac{GM}{2L} \cdot \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}.$$

Hence, assuming that the satellite is in the perihelion at $\varphi = 0$, we find by integration

$$\boldsymbol{v}(\varphi) = \boldsymbol{v}_{\max} - \frac{GM}{2L} \cdot \begin{pmatrix} \sin\varphi \\ 1 - \cos\varphi \end{pmatrix}.$$

The velocity vectors thus lie on a circle with center $\boldsymbol{v}_{\max} - \frac{GM}{2L}\boldsymbol{e}_2$ and radius $\frac{GM}{2L}$ (see Figure 1).

Depending on whether the satellites revolve around the Earth in the same direction (a) or in opposite directions (b), the angle between the two maximum (or minimum) velocities is α (a) or $\pi - \alpha$ (b) (see Figure 2). In each case, the circles on which the velocity vectors lie are overlapping. The maximal relative velocity, call it V_a or V_b, corresponds to the maximal distance between two points lying on each of the circles. With the designations given in the sketch, it can be calculated as follows:

$$\text{(a)}: \quad V_a = r_1 + r_2 + \sqrt{s_1^2 + s_2^2 - 2s_1s_2\cos\alpha}$$
$$\text{(b)}: \quad V_b = r_1 + r_2 + \sqrt{s_1^2 + s_2^2 + 2s_1s_2\cos\alpha}$$

As we have already seen, $r = \frac{GM}{2L}$. Moreover, using Equation (1) for the angular momentum, we can express the maximal velocity stated in Equation (2) as follows:

$$v_{\max} = \frac{GMm(1+e)}{l} = \frac{GM(1+e)}{2L}.$$

Hence,

$$s = v_{\max} - r = \frac{GMe}{2L}.$$

Finally, we obtain

$$\text{(a)}: \; V_a = \frac{GM}{2}\left(\frac{1}{L_1} + \frac{1}{L_2} + \sqrt{\left(\frac{e_1}{L_1}\right)^2 + \left(\frac{e_2}{L_2}\right)^2 - \frac{2e_1e_2\cos\alpha}{L_1L_2}}\right)$$

$$\text{(b)}: \; V_b = \frac{GM}{2}\left(\frac{1}{L_1} + \frac{1}{L_2} + \sqrt{\left(\frac{e_1}{L_1}\right)^2 + \left(\frac{e_2}{L_2}\right)^2 + \frac{2e_1e_2\cos\alpha}{L_1L_2}}\right)$$

In the case that $\alpha = \pi/2$, the maximal relative velocity is the same regardless of whether the satellites orbit in the same direction or not and equals

$$V = \frac{GM}{2}\left(\frac{1}{L_1} + \frac{1}{L_2} + \sqrt{\left(\frac{e_1}{L_1}\right)^2 + \left(\frac{e_2}{L_2}\right)^2}\right).$$

Furthermore, in the case that additionally it holds that $L_1 = L_2 =: L$,

$$V = \frac{GM}{2L}\left(2 + \sqrt{e_1^2 + e_2^2}\right).$$