Satellites

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§1 Specific angular momentum and areal velocity

Consider the situation of a satellite (of mass m) orbiting the Earth (of mass M). As we know from Kepler's first law, its trajectory is elliptical, with the earth in one of the ellipse's foci. Moreover, as the gravitational force acting on the satellite is centripetal, angular momentum is conserved. We thus define the *specific angular momentum* of the satellite as

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \frac{\mathbf{L}}{m}.$$

Since $\mathbf{r} = r\hat{\mathbf{e}}_r$ and $\mathbf{v} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta$, we obtain

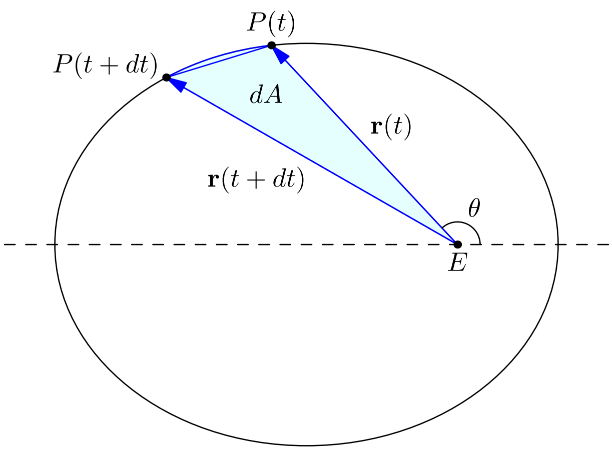
$$\mathbf{h} = \begin{vmatrix} \hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta & \hat{\mathbf{k}} \\ r & 0 & 0 \\ \dot{r} & r\dot{\theta} & 0 \end{vmatrix} = r^2\dot{\theta}\hat{\mathbf{k}}.$$

A related concept is that of *areal velocity*, i.e. the rate at which a line segment connecting a satellite and the Earth's centre sweeps out an area. By considering two very close points $P(t)$ and $P(t+dt)$ at which we can find the satellite,

$$dA = \frac{1}{2}|\mathbf{r}(t) \times \mathbf{r}(t+dt)| = \frac{1}{2}|\mathbf{r}(t) \times (\mathbf{r}(t+dt) - \mathbf{r}(t))| = \frac{1}{2}|\mathbf{r}(t) \times \mathbf{v}(t)|dt,$$

so

$$\frac{dA}{dt} = \frac{|\mathbf{r} \times \mathbf{v}|}{2} = \frac{h}{2} = \frac{r^2\dot{\theta}}{2}.$$



As h is constant, we have thus deduced (more or less rigorously) Kepler's second law, stating that the areal velocity is constant, and equal to half of the specific angular momentum.

§2 The velocity of an orbiting satellite

Let's analyze the motion of a satellite orbiting the Earth (counterclockwise) on an elliptical trajectory of eccentricity $\varepsilon < 1$. Newton's second law gives

$$\mathbf{a} = -\frac{GM}{r^2}\hat{\mathbf{e}}_r = -\frac{GM}{h}\dot{\theta}\hat{\mathbf{e}}_r = \frac{GM}{h}\dot{\hat{\mathbf{e}}}_\theta.$$

Integrating yields

$$\mathbf{v} = \frac{GM}{h}\hat{\mathbf{e}}_\theta + \mathbf{v}_0,$$

where $\mathbf{v}_0$ is a to-be-determined constant vector on integration. Considering the velocity of the satellite at the perigee and apogee respectively, we see that both $\mathbf{v}$ and $\hat{\mathbf{e}}_\theta$ have vertical direction, so $\mathbf{v}_0 = v_0\hat{\mathbf{j}}$. On the other hand, conserving the (specific) angular momentum, we have

$$\mathbf{h} = a(1-\varepsilon)\hat{\mathbf{i}} \times \left(\frac{GM}{h}\hat{\mathbf{j}} + v_0\hat{\mathbf{j}}\right) = -a(1+\varepsilon)\hat{\mathbf{i}} \times \left(-\frac{GM}{h}\hat{\mathbf{j}} + v_0\hat{\mathbf{j}}\right)$$

$$\iff (1-\varepsilon)\left(\frac{GM}{h} + v_0\right) = (1+\varepsilon)\left(\frac{GM}{h} - v_0\right).$$

Expanding the brackets will eventually give us $v_0 = \frac{GM\varepsilon}{h}$.

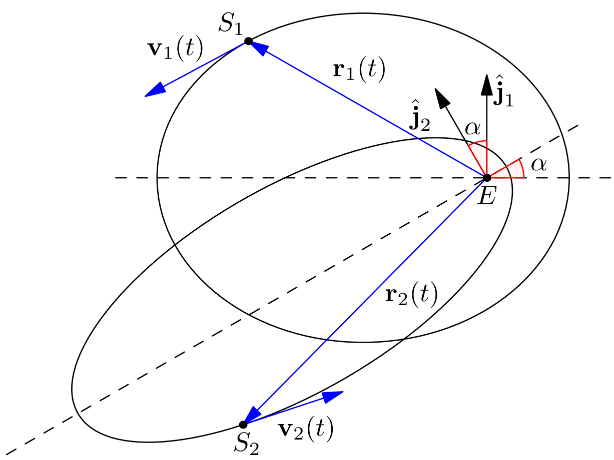
Therefore, we have calculated the velocity of the satellite as

$$\mathbf{v} = \frac{GM}{h}\hat{\mathbf{e}}_\theta + \frac{GM\varepsilon}{h}\hat{\mathbf{j}}.$$

§3 Maximum relative velocity

In our problem, we are given $L_{1,2}$ and $\varepsilon_{1,2}$. Also, $\frac{T_1}{T_2} \in \mathbb{R}\setminus\mathbb{Q}$. We will see later why this assumption is crucial (and how the problem's conclusion should have been better rephrased).

Take $\hat{\mathbf{j}}_1$ and $\hat{\mathbf{j}}_2$ be the two y-unit vectors specific to the two elliptical trajectories. The condition on $\hat{\mathbf{j}}_1$ and $\hat{\mathbf{j}}_2$ is just $\hat{\mathbf{j}}_2 = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\hat{\mathbf{j}}_1$.



Therefore the relative velocity is

$$\mathbf{v}_{\text{rel}} = \mathbf{v}_1 - \mathbf{v}_2 = \frac{GM}{2}\left(\frac{\hat{\mathbf{e}}_{\theta_1}}{L_1} - \frac{\hat{\mathbf{e}}_{\theta_2}}{L_2} + \frac{\varepsilon_1\hat{\mathbf{j}}_1}{L_1} - \frac{\varepsilon_2\hat{\mathbf{j}}_2}{L_2}\right).$$

By the triangle inequality,

$$v_{\text{rel}} = \frac{GM}{2}\left|\frac{\hat{\mathbf{e}}_{\theta_1}}{L_1} - \frac{\hat{\mathbf{e}}_{\theta_2}}{L_2} + \frac{\varepsilon_1\hat{\mathbf{j}}_1}{L_1} - \frac{\varepsilon_2\hat{\mathbf{j}}_2}{L_2}\right| \leq \frac{GM}{2}\left(\frac{1}{L_1} + \frac{1}{L_2} + \left|\frac{\varepsilon_1\hat{\mathbf{j}}_1}{L_1} - \frac{\varepsilon_2\hat{\mathbf{j}}_2}{L_2}\right|\right).$$

By the law of cosines, we finally arrive at

$$v_{\max} = \frac{GM}{2}\left(\frac{1}{L_1} + \frac{1}{L_2} + \sqrt{\frac{\varepsilon_1^2}{L_1^2} + \frac{\varepsilon_2^2}{L_2^2} - \frac{2\varepsilon_1\varepsilon_2\cos\alpha}{L_1 L_2}}\right).$$

For the particular version $L_1 = L_2 = L$ and $\alpha = 90°$, we get

$$v_{\max} = \frac{GM}{L}\left(1 + \frac{1}{2}\sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right).$$

We would normally end it here, but a question begs to be asked, namely if this maximum value can be achieved. Indeed, the equality case in the triangle inequality corresponds to when all (three) vectors are collinear and have the same orientation. So $\hat{\mathbf{e}}_{\theta_1}$ and $\hat{\mathbf{e}}_{\theta_2}$ must have the same orientation as $\frac{\varepsilon_1 \hat{\mathbf{j}}_1}{L_1} - \frac{\varepsilon_2 \hat{\mathbf{j}}_2}{L_2}$. So the maximum is achieved for exactly one position for each satellite.

This is where $\frac{T_1}{T_2} \in \mathbb{R} \setminus \mathbb{Q}$ comes in. Consider the natural bijection between a satellite's trajectory and $\mathbb{R}/T\mathbb{Z}$ (every point on the path corresponds to the moments in time $t_0, t_0+T, t_0+2T, \ldots$). Therefore, when satellite 1 finds itself at position P_0, corresponding to moments $t_0 + nT_1$, for $n \in \mathbb{Z}$, satellite 2 is at a position corresponding to $t_0 + nT_1 \pmod{T_2}$ (in the quotient group). The key is thus to understand the set

$$\{nT_1 \pmod{T_2} \mid n \in \mathbb{Z}\} \subset [0, T_2).$$

A theorem due to Kronecker states that

Theorem 3.1 (Kronecker)
Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then the sequence $(\{n\alpha\})_{n \geq 1}$ is dense in $[0, 1)$.

It's clear how it also implies that the above set is dense in $[0, T_2)$. What this basically says, is that when we find satellite 1 at a certain position P_0, satellite 2 can be almost anywhere, densely speaking. Therefore, when satellite 1 finds itself at the equality position for the maximum relative velocity, satellite 2 can get arbitrarily close to its equality position. Even though there are cases when both satellites can never both be at the equality position, their relative velocity can gen arbitrarily close to $v_{\max}$.

Thus, the more correct problem conclusion should have been to find the supremum of the relative velocity of the satellites, rather than the maximum.