Image of a circle

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§1 Euclidean constructions

Before we start to tackle the problem, we must have a good understanding of some of the main constructions that can be preformed solely with a straightedge and compass. Here are the main ones that we will use:

- Draw the perpendicular from a point to a line.
- Mark the center of a conic.
- Draw the tangents from a point to a conic.
- Draw the tangent at a point on a conic.

Though the series of steps required in order to obtain these objects is straightforward, the mathematical background is highly interesting. In what follows, I will present the proofs. The reader who wishes to skip this section may freely do so.

Theorem 1.1

Given a point P and line d, you can construct via straightedge and compass the perpendicular from P to d.

Proof. Consider a large enough circle of center P that intersects d at two distinct points A and B. Draw two circles with centers in A and B and of equal radii. Suppose they intersect at two distinct points C and D. Then by symmetry, CD is the perpendicular bisector of AB, and thus passes through P, making CD the desired line. □

Observe how from the above proof we have managed to also construct the midpoint of a segment and the parallel to a line through a point (hint: draw two perpendiculars).

Theorem 1.2

Given a conic Γ, you can mark its center via straightedge and compass.

Proof. Let AB and CD be two parallel chords of Γ. Let M and N be the midpoints of AB and CD respectively. The claim is that MN passes through the center of Γ.

To understand why this is the case, we can either look at the conic's respective equation in Cartesian coordinates, or make a more subtle argument, using some tools from projective geometry.

Let the two lines AB and CD intersect at P_∞. Since

$$(A, B; M, P_\infty) = (C, D; N, P_\infty) = -1,$$

MN is the polar of P_∞. However, P_∞ is on the line at infinity, which is the polar of the center O. By La Hire's theorem, O must be on the polar of P_∞, from where the conclusion follows.

Repeating the argument for another pair of parallel chords gives a second line on which O lies, finally constructing the center as their intersection. □

Theorem 1.3

Given a conic Γ and P a point not on Γ, you can construct the tangent from P to the conic via straightedge and compass.

Proof. Let two (distinct) lines through P intersect Γ at A, B and C, D. Let Q and R be the harmonic conjugates of P w.r.t. A, B and C, D (with the usual ruler and compass construction, considering a triangle $\triangle OAB$, taking a point $U \in OP$, taking $AU \cap OB = \{A'\}$ and $BU \cap OA = \{B'\}$ and finally intersecting $A'B' \cap AB = \{Q\}$). Since QR is the polar of P w.r.t. Γ, QR cuts the conic in the points where the tangents from P intersect Γ. □

Theorem 1.4

Given a conic Γ and P a point on Γ, you can construct the tangent in P to Γ via straightedge and compass.

Proof. A line through P cuts Γ a second time in Q. Consider R a random point on PQ outside Γ and let RA and RB be the tangents to Γ from R. Finally, let $AB \cap PQ = \{C\}$. Since $APBQ$ is harmonic (on the conic), the harmonic conjugate of C w.r.t. A, B is the point where the tangents from P and Q to Γ intersect. Thus after constructing D, PD is the sought-for line. □

§2 The physics of the lens and mathematical motivation

Consider a lens in the coordinate plane, with optical center O in the origin and main focus F with coordinates $(f, 0)$, where f is the focal length (making the x axis the principal axis). While we mostly care about what happens to the points in the $x \leq 0$ region, i.e. to the left of the lens, mathematically we can extend the resulting map to the whole projective plane.

Consider a random point P in the plane of coordinates (x, y) (with $x \leq 0$, $x \neq -f$). The image of P' of P through the lens is found by intersecting OP with FP_y, where P_y is the projection of P onto the y axis. With a little bit of algebra, we obtain the map

$$(x, y) \mapsto \left(\frac{xf}{x+f}, \frac{yf}{x+f} \right).$$

We also see that $x = -f$ causes some problems, as we know experimentally: light emitted from $(-f, 0)$ is refracted in a collimated beam, parallel to the principal axis. However, mathematically we can overcome all impediments, by generally considering the homography φ

$$(x : y : z)^T \mapsto \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 1 & 0 & f \end{pmatrix} (x : y : z)^T = (xf : yf : x + zf)^T.$$

This mathematical knowledge lets us make the following statements: lines map to lines, conics map to conics and lines tangent to a conic map to lines tangent to a conic, fact which may seem obvious physically, but nonetheless needs mathematical backup.

With this in mind, we try studying what happens to a circle Ω through such a map. Since in the proposed problem the image is real, we might as well assume Ω is to the left of $x = -f$. From now on, we denote by P' the image of a random point P.

Consider the inscribed square $ABCD$ with AC parallel to the y axis. Denote by d_A, d_B and so on, the tangents in A, B and so on, to Ω. Call $\Gamma = \Omega'$ the image of the circle through the lens. Observe that d'_B and d'_D are the tangents in B' and d' to Γ, and since initially they are parallel to the lens, their image stays parallel to the lens. On the other hand, BD is perpendicular to the lens, so its image $B'D'$ passes through F. However, since d'_B and d'_D are pairs of parallel tangents to Γ, the line formed by the tangency points, namely $B'D'$ must pass through its center (for convenience we will denote it by O', even though it's not the image of the optical center, nor the image of the center of Ω). In order to verify the latter fact, try making an argument similar to the one in the proof of Theorem 1.2.

All of this work has given us that F, O', B', D' are collinear, and the tangents in B' and D' to Γ are perpendicular to the principal axis.

Now we inspect A and C. d_A and d_C are perpendicular to the lens, so d'_A and d'_C are the tangents from F to Γ. Moreover, A' and C' are the tancengy points. Since $A'B'C'D'$ is harmonic (on Γ), $A'C'$ is redundantly parallel to d'_B and d'_D, however we know that already by physical considerations.

Now we introduce O. A, O, A' and C, O, C' are collinear, and $AC \parallel A'C'$. Thus, let's consider the homothety of center O mapping AC to $A'C'$. Points B and D are mapped to, say, U and V respectively such that $A'UC'V$ is a square. By homothety, B, O, U and D, O, V are collinear. But so are B, O, B' and D, O, D', so anyways, $\{O\} = UB' \cap VD'$.

We seem to be good to go. Only, there is one thing left to properly understand at the end: when we construct square $A'UC'V$, which way do we need to construct it? In other words, which one is O? $UB' \cap VD'$ or $VB' \cap UD'$? Though subtle, the distinction must be made, since either way the point of intersection is on the principal axis. The way to go is like this: construct the square such that U and B' are in different half-planes determined by $A'C'$. It's quite easy to understand why. Since F and Γ are in the same half-plane determined by the lens, if d'_D is closer to F than d'_B, d_D must be further away from the lens than d_B, and thus D is further than B. Through homothety however, the order relation is maintained (since the homothety is of negative ratio), so V is further than U from the lens.

Finally, a good question is why for any ellipse (notice I didn't say "conic") there exists a unique pair of lens and circle, such that the circle is mapped to the ellipse through the lens. Well if we construct step by step the points described above (obviously, there is only one way to do so), we will arrive at points O, A, B, C and D, with the last four forming a square. If we are lucky enough for the rest of the points on Γ to land on the circumscribed circle of $ABCD$ through the inverse of the homography, then we are done. But is it always the case? Well, the inverse homography is also a homography, so $\varphi^{-1}(\Gamma)$ is a circumconic of $ABCD$. We know that it is also tangent to the parallels to the lens through B and D, and to the parallels to the principal axis through A and C. By a deep fact from algebraic geometry, a conic is uniquely determined by n points on it and $5 - n$ lines to which it is tangent (actually we don't need the full power of algebraic geometry, as the result for conics has a simple enough proof in [1]). Anyways, since the circumcircle of $ABCD$ verifies these constraints and there is at most one such conic, $\Omega = (ABCD)$ must coincide with $\varphi^{-1}(\Gamma)$.

§3 The solution

I will freely use the shortcuts discussed in the first section. I will keep the primed notation for a better link with the observations made above.

- Construct the center O' of Γ.
- Intersect FO' with Γ in B' and D' (for instance, D' between F and B').
- Let the tangents from F to Γ intersect the conic in A' and C'.
- Take the midpoint of $A'C'$ and its perpendicular bisector.
- Construct the circle of diameter $A'C'$ and intersect it with the perpendicular bisector in U and V such that U and B' are in different half-planes determined by $A'C'$.
- Mark $\{O\} = UB' \cap VD'$.

We have thus found O. For the proposed configuration, the lens's center has coordinates

$$O = (-1.12358, -0.58132).$$