Physics Cup Problem 2

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1 Some Observations

First, we note that the process of taking an image under a lens satisfies the following conditions.

1. Straight lines are preserved as straight lines.
2. Images of all lines parallel to the optical axis pass through the focal point.
3. Lines parallel to the lens orientation will stay parallel to the lens orientation.
4. Lines passing through the center of the lens are left unchanged.
5. The points where the images of two perpendicular lines cross the focal plane will create a right triangle with the center of the lens.

2 Proof

1. The formula for translating a point (x, y) to its image (x', y') when all coordinates are measured from the center of the lens with the x-axis coinciding with the optical axis is $(x', y') = (fx/(x-f), -fy/(x-f))$. When we take the image of a line, we have a dependence of $y = mx + b$ with m, b being constants. Then we obtain $y' = -(m + b/x)x'$ which turns to $y' = -(m + b/f)x' + b$. Therefore, the image is a straight line.
2. When $m = 0$, the above equation turns to $y' = -bx'/f + b$ which contains the point $(f, 0)$ for all values of b.
3. When $m = \infty$, we have a line characterized by $y' = -\infty x'$, thus it is also oriented the same way.
4. When $b = 0$, we obtain $y' = -mx'$ which means the image of the line passing from the center is same as its continuation to the plane lying on the other side of the lens. A sign change is introduced in the equation because the directions where x and x' increase are opposite yet y and y' increase in the same direction.

5. Let there be two perpendicular lines $y_1 = mx + b_1$ and $y_2 = -x/m + b_2$. Their images will be $y_1' = -(m+b_1/f)x' + b_1$ and $y_2' = (1/m - b_2/f)x' + b_2$. They will cross the focal plane at $x' = f$. At that point we have $y_1' = -mf'$ and $y_2' = f'/m$. The angle between the first point and the x-axis is $\tan\theta_1 = -m$ and for the second point $\tan\theta_2 = 1/m$. Thus, we have $\tan\theta_1 \tan\theta_2 = -1$ which implies $|\theta_2 - \theta_1| = \pi/2$.

3 Solution of the Problem

Now, we can start the solution by observing that the tangent lines passing from the focal point close to the image and boxing the ellipse are images of lines parallel to the optical axis that are tangent to the original circle.

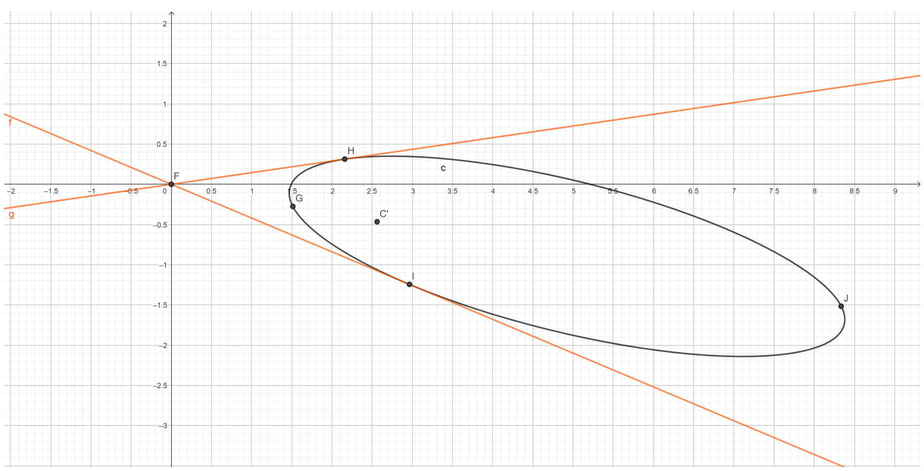

Figure 1: Images of tangent lines to the circle parallel to the optical axis.

Points H and I in the image are the lowermost and uppermost points of the circle measured according to the lens orientation. Thus, the line passing from H and I will give us the lens orientation.

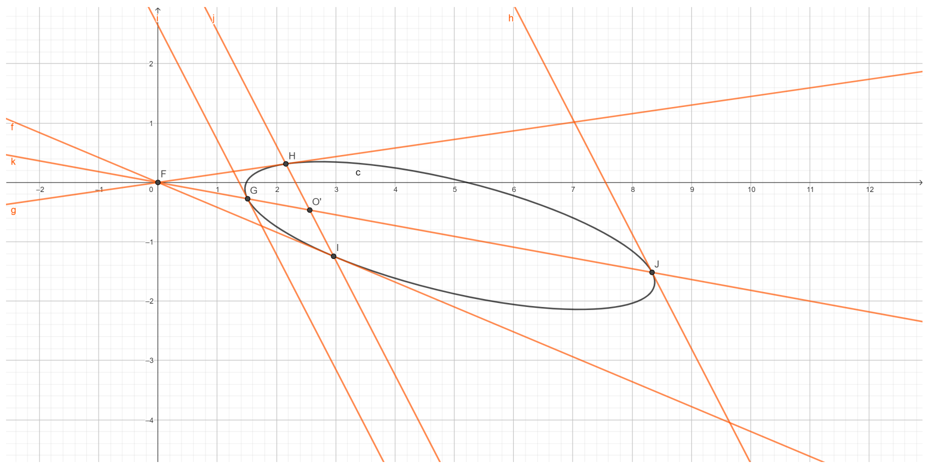

Figure 2: Construction of the images of points closest and farthest away from the lens.

The lines parallel to the $\overline{HI}$ that are tangent to the ellipse give us the images of points that are closest and farthest from the lens J and G respectively. The point O' is the image of the center of the circle.

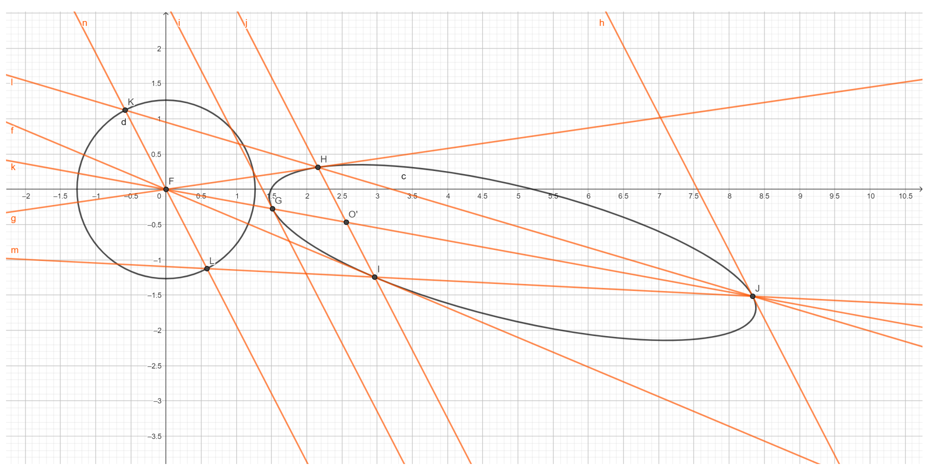

Figure 3: Construction of two points on the focal plane creating a right triangle with the center of the lens.

Thus the lines $\overline{HJ}$ and $\overline{IJ}$ are the images of two perpendicular lines. When we connect the points where they meet the focal plane with the center of the lens, they will make a right angle by observation 5. Therefore, the center of the lens will be located on the circle centered on the midpoint between K and L which passes from both points. In this problem, midpoint coincides with F.

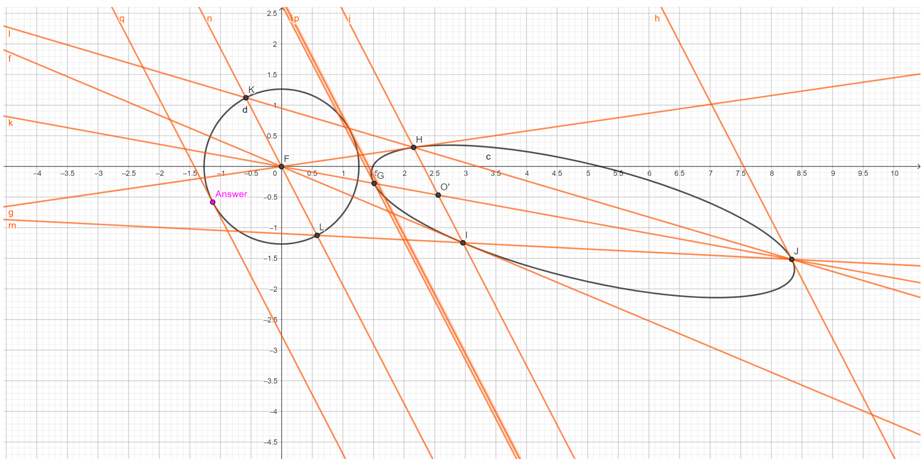

Figure 4: Finding the center of the lens.

Since we know that the line connecting the center of the lens and the focal point (optical axis) has to be perpendicular to the lens, we can find it as the intersection of the circle and tangent line drawn to the circle that is parallel to the lens orientation. There are two candidate points for being the center, yet we know that F is the focal point that is closer to the ellipse, thus the one on the left has to be the real center. It has coordinates (-1.12358, -0.58132). There might be some minor discrepancies because GeoGebra cannot perfectly identify intersections.