Physics cup 2024 Problem 2

Val Karan

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1 Introduction

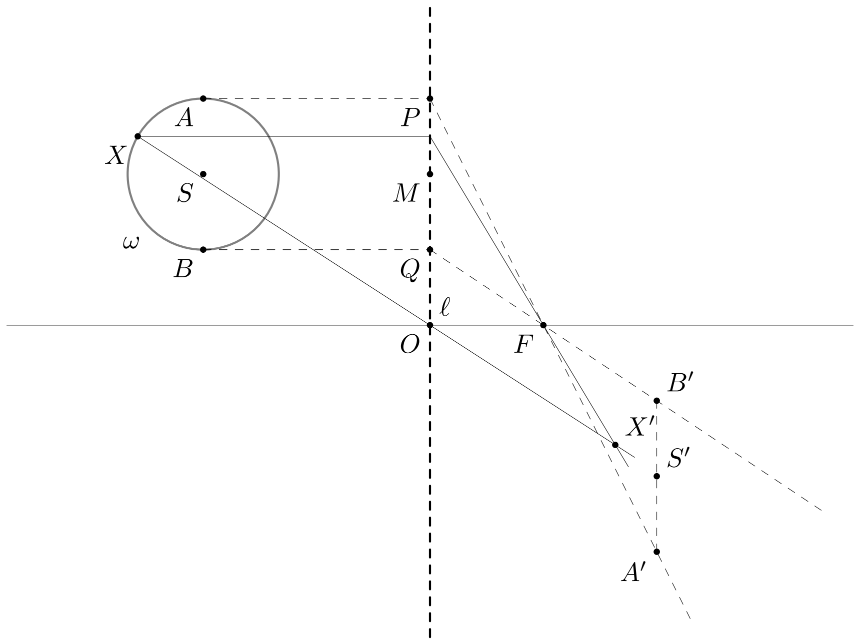
The solution is organized as follows: First I derive a few lemmas concerning images of circles produced by an ideal thin lens. Then I use these lemmas and information in the given Geogebra file to construct the exact location of point O, the centre of the lens. The construction itself is also presented in another Geogebra file.

2 Opening remarks

We are assuming that we are working within the approximation of geometrical optics, i.e. light can be treated as rays, and wave effects other that refraction and reflection can be neglected. The lens that we are working with is an ideal thin lens, i.e. any ray parallel to the optical axis bends at the lens and goes through point F (the focus) and any ray that goes through point O does not bend (refract). We are also assuming that the problem is completely two dimensional.
Throughout this solution the point X' represents the image of point X produced by the lens.

3 Lemmas

Consider an ideal thin lens with centre O and focal point F. Let ℓ (the line representing the lens) be the line through O perpendicular to OF, the optical axis. Consider a circle ω with centre S. Assume this circle has a real image produced by the lens that is an ellipse ω'. Consider a point $X \in \omega$. Let $p(X)$ be the orthogonal projection of X onto ℓ. Let $d(X)$ be the directed distance (the coordinate along ℓ) between O and $p(X)$.
Lemma 1: Let $A, B \in \omega$ be such that A has the maximal value of d and B has the minimal value of d (out of all points on ω). If we were to take ℓ to be the y axis, these would be the top and bottom points of the circle. Then FA' and FB' are tangent to the ellipse ω'. Furthermore, the midpoint of segment $\overline{A'B'}$ is just S'.
Proof: Consider the following sketch:



Let P, Q, M be the projections of points A, B, S on to line ℓ respectively. Note that M is the midpoint of $\overline{PQ}$. Any ray of light parallel to the optical axis OF coming from some arbitrary point $X \in \omega$ will hit the lens in some point on the segment $\overline{PQ}$. Therefore, its image X' must lie on a line that is between the lines PF and QF. Thus the whole image ω' of the circle lies necessarily in the region between these lines. Note that points A', B' lie on lines PF, QF respectively and are thus on the boundary of said region, they are also the only points in ω' that have this property. This implies that A', B' are the tangent points of F with respect to the ellipse ω'.
The points A, B, M are equidistant from the lens implying that their images A', B', S' are as well. This means that $A'B' \parallel \ell$ and $S' \in A'B'$. Consider the homothety transformation with F as its centre and scaling factor $k = -\frac{|A'F|}{|PF|}$. Since $A'B' \parallel PQ$ we have that during this homothety the triangle $\triangle PQF$ gets mapped to triangle $\triangle A'B'F$. The midpoint M of $\overline{PQ}$ gets mapped to $MF \cap A'B'$ which is also the midpoint of $A'B'$ (by homothety properties), but this point is by definition just S' (since S' also lies on $A'B'$). Thus S' is the midpoint of $\overline{A'B'}$.

Lemma 2: Consider points A, B as defined in Lemma 1, but also points C, D being the closest and the farthest point of ω from the lens respectively. Let $E' = A'C' \cap B'D'$. Then $|FE'| = |FO|$.
Proof: Although there is a standard geometric proof for this claim, I will be presenting a much shorter and more elegant proof here using projective geometry. First notice that lines AC and BD are parallel and both form an angle of 45° with line ℓ. Thus their "intersection" is the point at infinity E such that EY makes an angle of 45° with ℓ for any $Y \in \ell$. The image of this intersection is just the intersection of images which is just E'. Let a, b, f be the distances of E, E', F from ℓ respectively. By the ideal lens equation (and the fact that E is infinitely distant) we have:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{f} \Longrightarrow \frac{1}{b} = \frac{1}{f} \Longrightarrow b = f$$

From this we have that E' lies on the parallel through F with respect to ℓ. We also know that E, O, E' are collinear and $\sphericalangle(EO, \ell) = 45°$. This means that $\triangle OFE'$ is a right triangle with an inner angle of 45°. This implies $|FE'| = |FO|$.■

4 Constructing O

From now on I will be using notation that is present in my Geogebra file, but it will become clear how these new points are related to points discussed in Lemmas 1 and 2. We will first construct points $G, H \in c$, the tangent points of F with respect to the given ellipse c. Notice that by Lemma 1, these correspond to points A', B'. Thus their midpoint I corresponds to S'. The ray of light parallel to the optical axis from points C, D is the exact same as for point S and so $S'F$ is the same line as $C'F$ and $D'F$. The intersections of line IF with c then correspond to C' and D', let us denote these intersections as J, K. Note that the optical axis OF must be perpendicular to line HG, as we have seen in the proof of Lemma 1. Thus the optical axis is just the perpendicular through F on line HG, let us denote it by j. Notice that $L = HJ \cap GK$ corresponds to E'. From Lemma 2 we have $|FL| = |FO|$. By drawing a circle d with centre F and radius $|FL|$, and intersecting it with j we obtain point O (since O is necessarily on the optical axis j). Doing this in Geogebra gives the following coordinates: $\boxed{O(-1.12358, -0.58132)}$. This construction is also depicted in Figure 1.

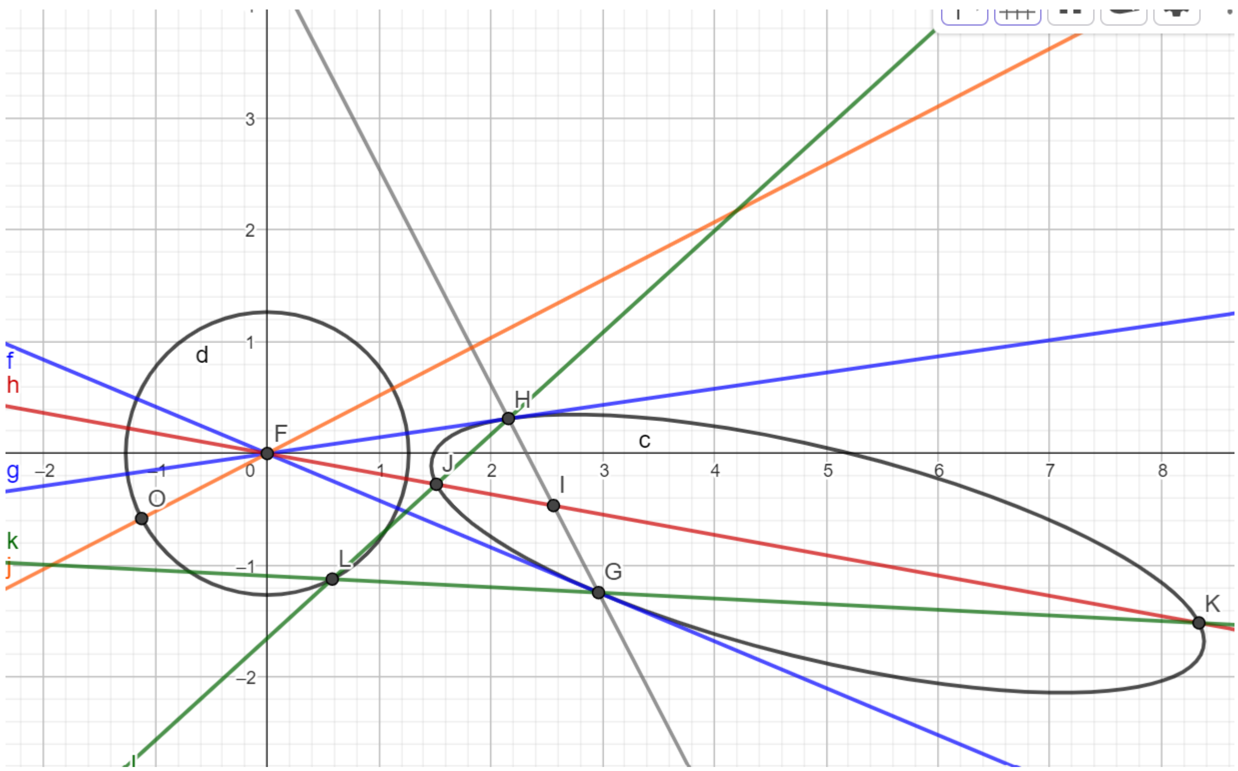


Figure 1: Constructing point O