# Problem 2 – Image of a circle

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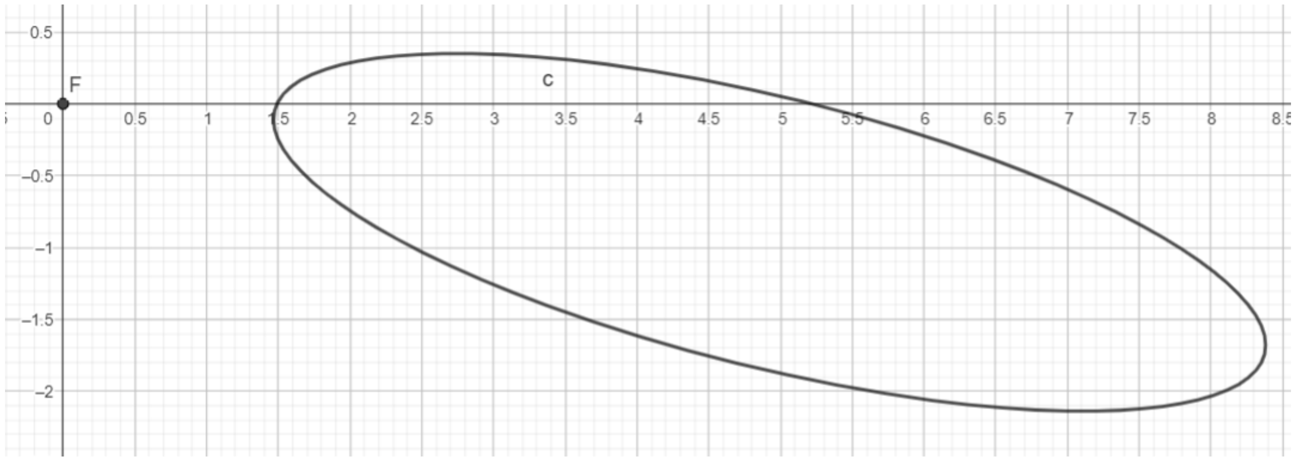
## Task

In the figure below, the ellipse is a real image of a circle created by an ideal thin lens. The point $F$ is the closer-to-the-ellipse main focus of the lens. Optical axis lies in the plane of this figure. Construct geometrically the centre $O$ of this thin lens.




## Structure of this document

On the next page two sketches are shown that illustrate the imaging process. These are then used to motivate the procedure applied in Geo Gebra.

## Sketches

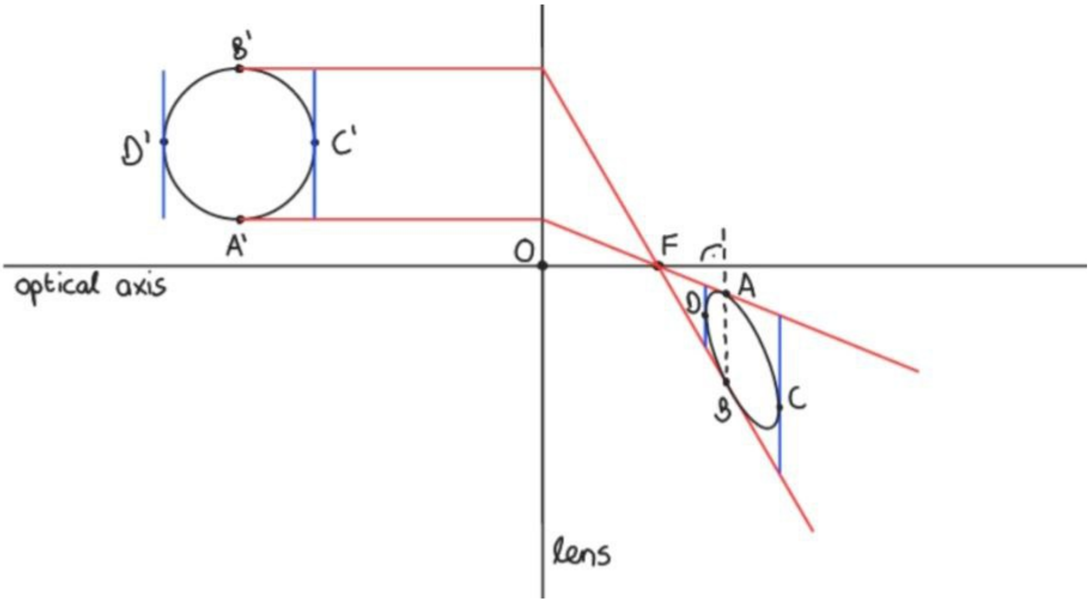



Figure 1: Rays tangent to the circle are also tangent to the ellipse after refraction. Moreover, tangent lines to the circle are mapped to tangents to the ellipse.

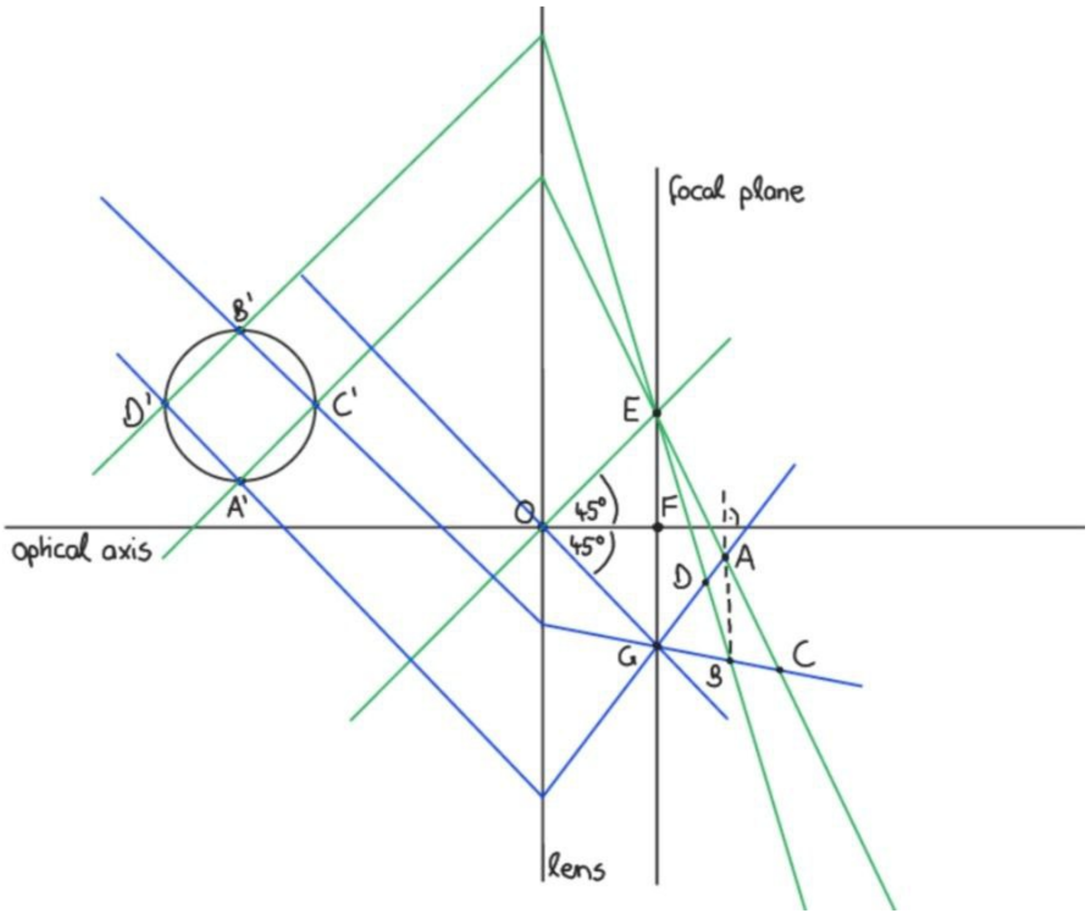



Figure 2: Parallel incident rays converge to a point in the focal plane.

## Procedure in Geo Gebra

In the following, the constructions carried out in Geo Gebra are described and it is explained with the help of the above sketches why the point $O$ found in this way represents the centre of the thin lens.

| Objects drawn in Geo Gebra | Designation | Physical motivation |
|---|---|---|
| Both tangents to the ellipse that run through $F$<br><br>Contact points of the tangent lines on the ellipse | $f$ and $g$ (orange)<br><br>$A$ and $B$ | The two rays of light that originate tangentially from the circle at the points $A'$ and $B'$ and hit the lens perpendicularly run along the orange straight lines drawn in Geo Gebra after refraction (see Figures 1 and 3). |
| Line through $A$ and $B$<br><br>Straight line parallel to $h$ passing through $F$ and<br><br>Perpendicular line passing through $F$ | $h$ (blue)<br><br>$i$ (blue)<br><br>$j$ (red) | Just as the line $A'B'$ is perpendicular to the optical axis, so is $h = AB$. This makes it possible to draw the focal plane ($i$) and the optical axis ($j$). |
| Tangent lines to the ellipse that run parallel to $h$<br><br>Contact points of the tangents on the ellipse | $k$ and $l$ (green)<br><br>$C$ and $D$ | The images of the two straight lines that run perpendicular to the optical axis and touch the circle at the points $C'$ and $D'$ are tangent lines to the ellipse that are also perpendicular to the optical axis (see Figure 1). |
| Lines through $A$ and $C$ as well as through $B$ and $D$<br><br>Point of intersection (on $i$)<br><br>Lines through $A$ and $D$ as well as through $B$ and $C$<br><br>Point of intersection (on $i$) | $m$ and $n$ (brown)<br><br>$E$<br><br>$p$ and $q$ (brown)<br><br>$G$ | Parallel incident rays converge to a point in the focal plane. The two rays of light travelling through either $A'$ and $C'$ or $B'$ and $D'$ intersect at point $E$ after refraction. The two rays that pass through $A'$ and $D'$ or $B'$ and $C'$ converge to the point $G$. As the images of the points lie on the refracted rays, it is possible to construct $E$ and $G$ from them (see Figure 2). |
| Circle with centre in $F$ running through $E$ and $G$ | $d$ (black) | The two parallel ray bundles that converge to $E$ or $G$ both hit the lense at an angle of 45°. As rays passing through the center of the lens are not refracted, $\angle GOE = 90°$ (see Figure 2). Consequently, $O$ lies on the circle with diameter $\overline{EG}$. |
| Intersection of $d$ and $j$ (so that $i$ separates $O$ and the ellipse) | $O$ | $O$ is also located on the optical axis and can therefore be identified as the intersection of the optical axis ($j$) with the circle ($d$) lying to the left of the focal plane. |

This gives $O = (-1.12358, -0.58132)$.

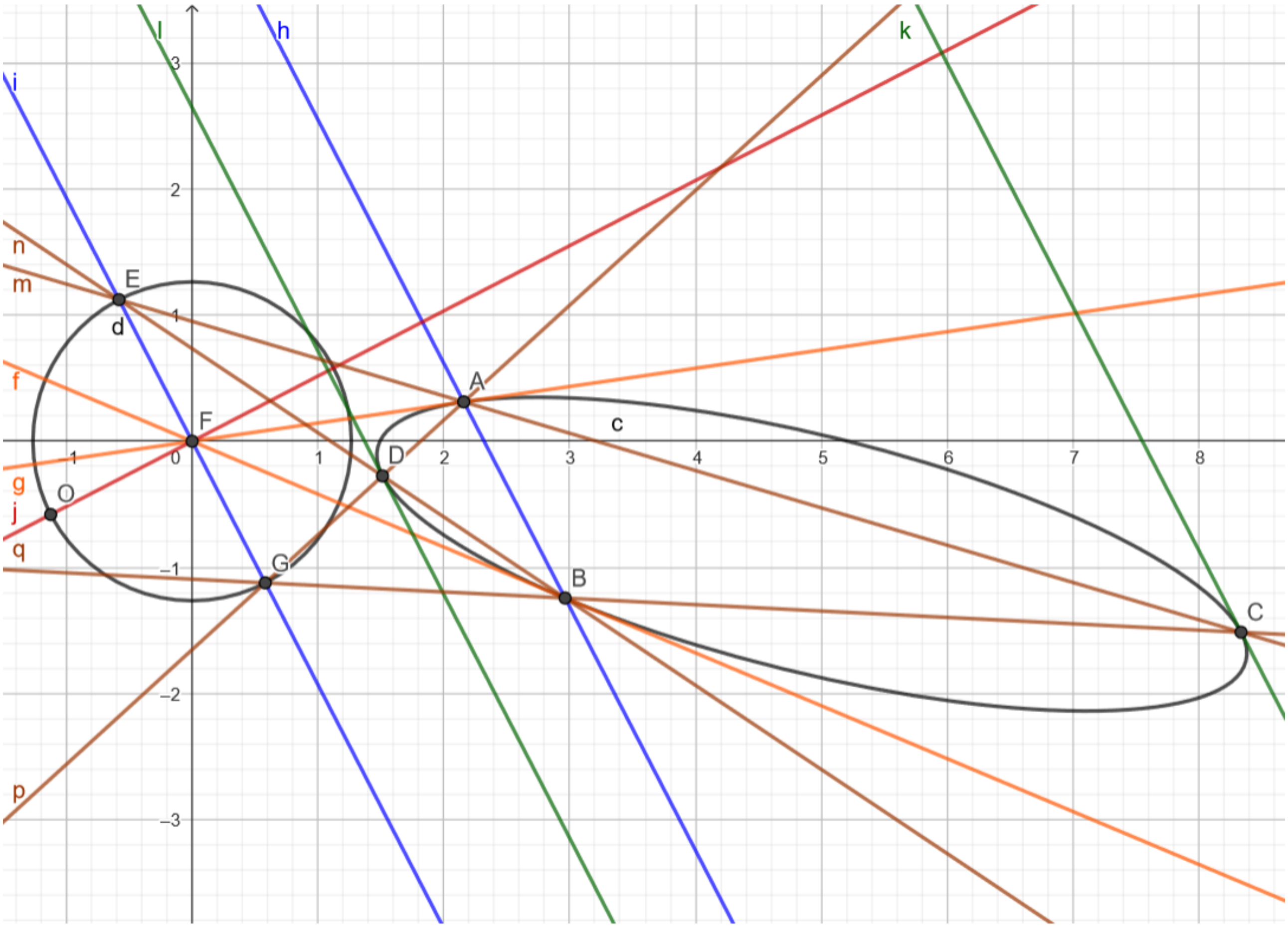



Figure 3: Constructions in Geo Gebra