problem 2

December 6, 2023

1 Solution

We first prove a lemma first.

Lemma: Straight lines are mapped to straight lines by ideal thin lenses

Proof: imagine a ray of light going along a given line l_1. l_1 would be bent in some direction by the lens, but it would travel along a straight line l_2 after passing through the lens. The image of all points on l_1 lies on l_2.

Note that lines perpendicular to principal axis are imaged into lines perpendicular to principal axis.

Let's call the image of any point X, made by the lens, to be X'. Note that $(X')' = X$.

Consider tangents from the focus F to the given ellipse. Those lines would be imaged into lines parallel to the principal axis of the lens. Also since they touch the ellipse(i.e. they are tangent to it), their images would touch the circle ω whose image is the given ellipse. Let's call the points of tangency of the tangents to ellipse from F to be G and H. Then G' and H' are points of tangency of lines parallel to principal axis to ω. Hence the line $G'H'$ must be perpendicular to the principal axis. So GH is also perpendicular to the principal axis. Construct a line perpendicular to GH passing through F and that gives us the principal axis of the lens.

Note that points that lie on a line perpendicular to principal axis are present at uniformly magnified distances from principal axis in the image and hence the ratio of distances is preserved.

Let the midpoint of GH be I and the centre of the ellipse be J. Then I' represents the centre of ω. Also J' lies on a line parallel to principal axis of lens and passing through I' because J is the midpoint of a chord perpendicular to principal axis(actually it is the midpoint of all chords of ellipse passing through it). Thus the image of line IJ (i.e. line $I'J'$) is a line parallel to principal axis. Call the points where IJ intersects the ellipse to be K, L. Then the tangents at K', L' to ω must be perpendicular to principal axis. So tangents at K, L to the ellipse are also perpendicular to the principal axis of the lens. Call the points of intersections of these tangents with principal axis O, P.

Let $FP = x$, $FO = y$. Let the focal length of lens be f. Let the slope of lines FG and FH with respect to principal axis of lens be $\tan(\theta_1), \tan(\theta_2)$ respectively. Since FG and FH become parallel to the principal axis after passing through the lens, the distance between them after refraction is

$$f(\tan(\theta_2) - \tan(\theta_1))$$

and that will be the diameter of ω.

Now we use newton's thin lens formula ($u_{F_1} v_{F_2} = f^2$). So that distance between K' and L' along principal axis is

$$\frac{f^2}{y} - \frac{f^2}{x} = \frac{f^2(x-y)}{xy}$$

and this is also equal to the diameter of ω. Equating the 2 expressions, we get that $f = x(\tan(\theta_2) - \tan(\theta_1)) \cdot \frac{y}{x-y} = MN \cdot \frac{y}{x-y}$ where M, N are points of intersection of tangent to ellipse at L with FG, FH respectively. Now we just need to find a point T on line MN such that $MT/MN = FO/OP$ and that can easily be done by Thales theorem. Make any arbitrary line passing through M and find points on it R, S so that $MR = FO$ and $MS = OP$ using compass tool. Then join S with N and draw a parallel line to SN passing through R and let it intersect MN at T which is the desired point. Now just draw a point at the same distance as length MT to F on the farther side of ellipse and that gives us the centre of the lens.

Carrying it out on geogebra gives centre of lens $U = (-1.12358, -0.58132)$