Physics cup 2024 problem 2

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To begin with a personal note, I started this problem in complete confusion, for 6 hours I was searching for resources, toying around with desmos, and eventually realizing everything I wrote so far was complete nonsense. I knew this problem will be my new obsession for at least the next two weeks, as there was so much to find and yet so little at my disposal. Then I woke up with all the answers:

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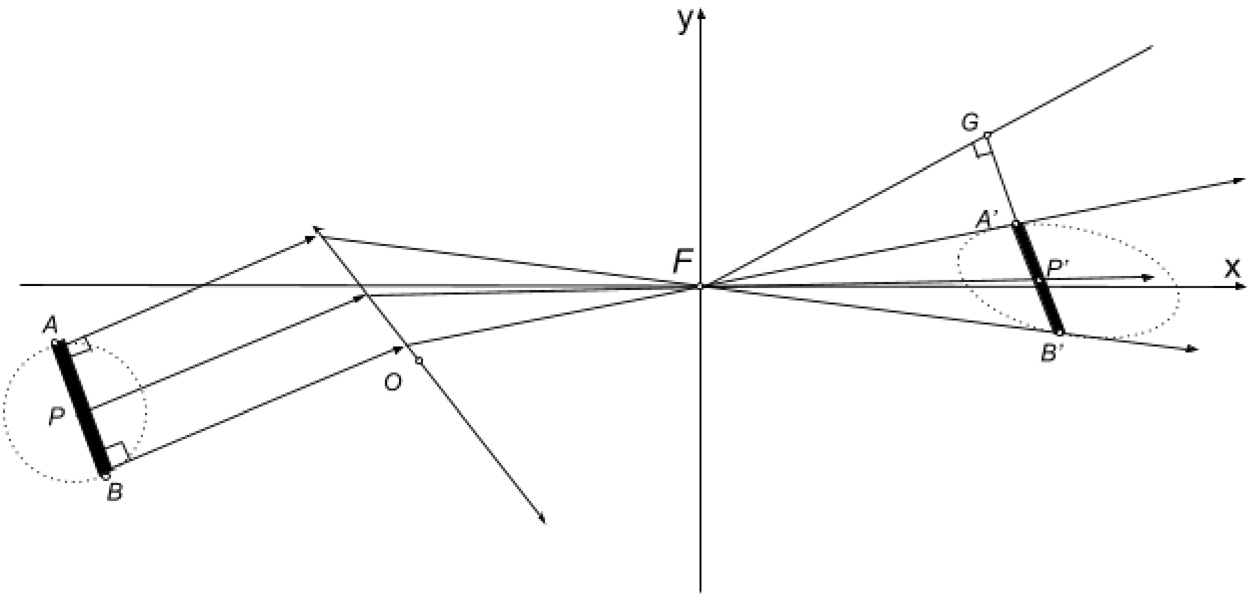
Let's look at the two points in the circle which the tangent is the optical axis (*A* and *B*). All images form with the intersection of two rays, and all the rays coming from the object and parallel to the optical axis will be in between the rays of said two points, and so the resulting image will be confined between two said rays. Moreover, since the image of the aforementioned two points lies on these two rays, the rays are the tangents of the ellipse from the focal point.

If we now look only at the line between those points (which we name *A'* and *B'*),



Because the tangents of the point are parallel to the optical axis (and each other), the line between them is the diameter of the circle (which is perpendicular to the optical axis. The image of a line perpendicular to the optical axis of a thin lens is a scaled line parallel to the original one, since the scaling factor is constant we can say two things:

- The image of the center of the circle is the midpoint of the contacts with the tangents to the ellipse
- The chord of contact from the focal point is perpendicular to the optical axis of the lens

To find the radius a of the circle, we will use the lens formula:

$$\frac{1}{u} + \frac{1}{v_1} = \frac{1}{f}$$

$$u = \frac{v_1 f}{v_1 - f}$$
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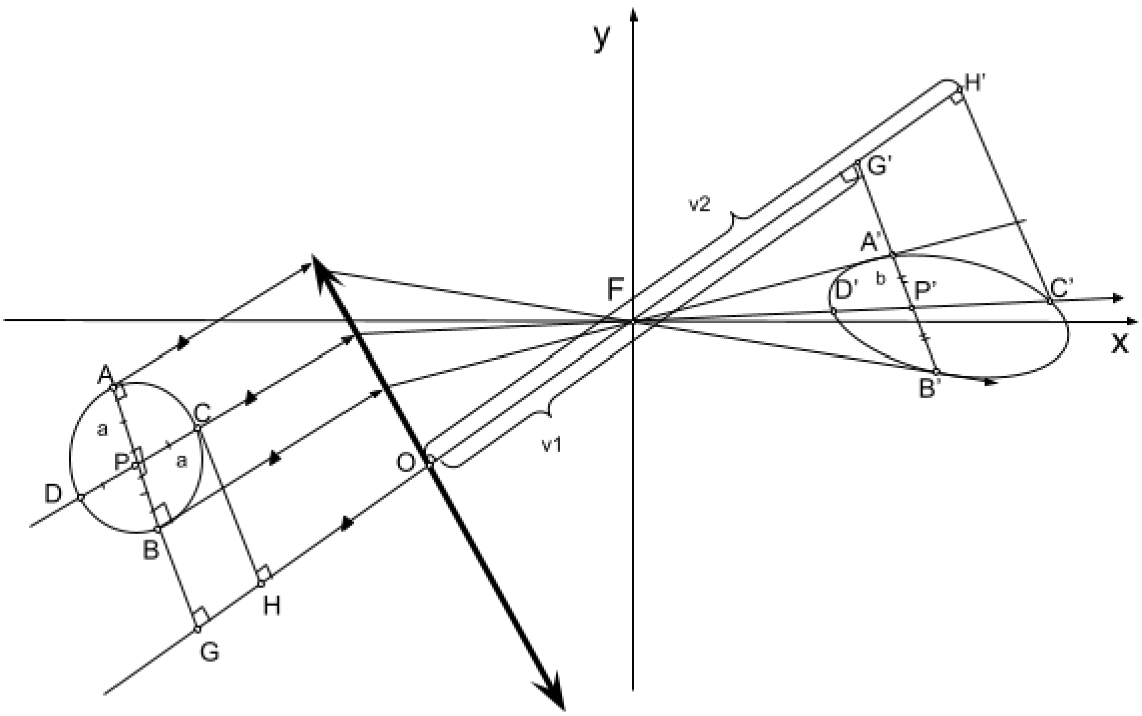
$\frac{u}{v} = \frac{a}{A'P'}$

$a = \frac{A'P'\cdot f}{(v_1-f)} = \frac{A'P'\cdot f}{(v_1-f)}$

The ray passing through *P* at the left side intersects with two points on the circle, hence the images of these two points at the right side must be on the same ray. Since that ray is parallel to the optical axis, it also passes through the focal point in the right side. The line from *P'* to the focal point intersects with the ellipse twice. Therefore, the points of intersection between *P'F* and the ellipse (*C'* and *D'*) correspond to the end points of the diameter of the circle which is parallel to the optical axis (*C* and *D*). (that was extremely wordy, here is a picture)



Now let's find the focal point using our new points

$\frac{1}{OG} + \frac{1}{OG'} = \frac{1}{f} = \frac{1}{OF}$

$OG = \frac{OG'\cdot OF}{OG'-OF} = \frac{v_1 f}{v_1-f}$

$\frac{1}{OH} + \frac{1}{OH'} = \frac{1}{f} = \frac{1}{OF}$

$OH = \frac{1}{u_1} = \frac{|OH'|\cdot|OF|}{OH'-OF} = \frac{v_2 f}{v_2-f} = OG - GH = OG - a = \frac{v_1 f}{v_1-f} - \frac{A'P'\cdot f}{(v_1-f)} = \frac{v_1 f}{v_1-f} - \frac{bf}{v_1-f} = \frac{(v_1-b)f}{v_1-f}$

$\frac{v_2 f}{v_2-f} = \frac{(v_1-b)f}{v_1-f}$

$\frac{v_2}{v_2-f} = \frac{(v_1-b)}{v_1-f}$

$v_2(v_1 - f) = (v_1 - b)(v_2 - f)$

$v_1 v_2 - v_2 f = v_1 v_2 - v_2 b - v_1 f + bf$

$- v_2 f =- v_2 b - v_1 f + bf$

$v_2(b - f) = f(b - v_1)$

$v_2 = \frac{f(b-v_1)}{b-f}$

$v_2 - f = \frac{f(b-v_1)}{b-f} - f = \frac{f(b-v_1)-bf+f^2}{b-f} = \frac{-v_1 f+f^2}{b-f} = \frac{f(v_1-f)}{f-b}$

We know both b $(b = A'P')$, $v_1 - f$ $(v_1 - f = FG')$ and $v_2 - f$ $(v_2 - f = FH')$.

$\frac{v_2-f}{v_1-f} = \frac{f}{f-b}$

$\frac{v_2-f}{v_1-f}f - \frac{v_2-f}{v_1-f}b = f$

$f = \frac{\frac{v_2-f}{v_1-f}b}{\frac{v_2-f}{v_1-f}-1} = \frac{(v_2-f)b}{(v_2-f)-(v_1-f)} = \frac{FH'\cdot A'P'}{FH'-FG'} = \frac{FH'\cdot A'P'}{G'H'} = \frac{FP'\cdot A'P'}{C'P'}$

To get f geometrically, we will use the Talos principle:

1- Construct a circle of radius *FP’* on an arbitrary point *K*

2- Construct another circle of radius *FC’* around point *M*, which is at the perimeter of the circle at *K*

3- Construct a line between M and K, the intersection point which lies within the circle at *K* we call *L*

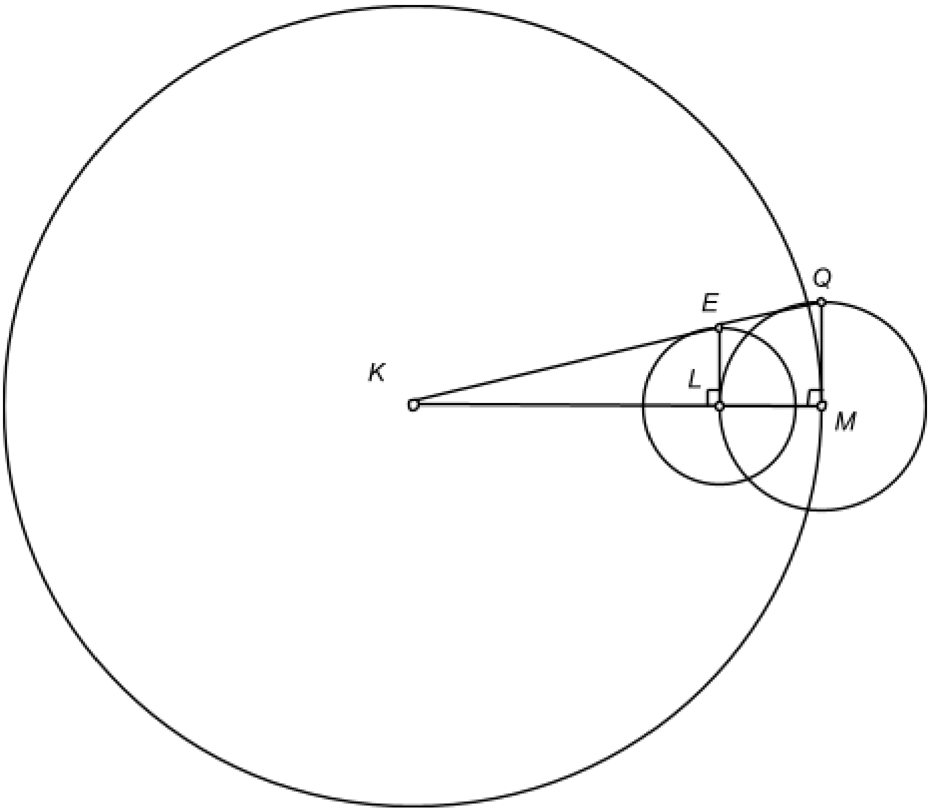
4- Make a circle of radius *A’P’* around point *L*

5- Construct a perpendicular line to *MK* at *L*, the intersection point between the line and the circle around *L* we call *E*

5- Construct a perpendicular line to *MK* at *M*

6- Construct a line from *E* to *K*, the intersection point between said line and the line from 5 we call *Q*

7- *MQ* is our focal length



8- Make a line perpendicular to *A’B’* and place it on *F*, this is our optical axis

9- Construct a circle of radius *MQ* at point *F*. The intersection point between said circle and the line from 8 is the position of the lens!

We are not completely done, as there are two intersection pointsm, but since we know the focal point is the one closer to the ellipse we can rule out the placement further from the ellipse and remain with:

(-1.12358,-0.58132)

For the skeptics amongst us, we can do the same with points *D* and *D'* and reach the same result.

Note: size doesn't matter

Some of you might be thinking: "what if the lens has too small of a diameter for rays perpendicular to the optical axis to reach it? That would make the tangents from the focal point to the ellipse have no meaning and the problem is ruined".
The answer to the question is: "it doesn't matter".
Because $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ holds regardless of diameter, each point of the circle will be mapped to the same ellipse regardless of the diameter of the lens, making the problem for a lens with a small diameter and the problem for a lens with large diameter equivalent.