

Problem 5 – Water pipe

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Task

A water pipe of length L and of internal radius r runs under ground and is surrounded over its entire length by soil at temperature T_0 . The specific heat of the soil is c , the density is ρ , and the heat conductance is κ . The characteristics of the pipe walls are identical to those of the soil. A boiler supplying water at a constant temperature $T_1 > T_0$ is attached to the inlet of the pipe, and at the moment of time $t = 0$, a tap is opened at the outlet of the pipe. The water in pipe starts flowing at a constant speed v . How long one has to wait for the water flowing out of the tap to become warm if it is known that this waiting time is significantly bigger than L/v ? We call water warm if its temperature is higher than $\frac{T_1+T_0}{2}$. Only an estimate of the answer is required: you need to provide the correct functional dependence of the waiting time while estimating the magnitude of its parameters. The specific heat of water is c_w , and its density is ρ_w ; the water flow in the pipe is turbulent so that the water temperature can be assumed to be constant over a cross-section of the pipe.

Solution

Let us call the waiting time τ and let us define $T_{1/2} = \frac{T_1+T_0}{2}$. Our aim is to find the temperature of the water $T_w(l, t)$ as a function of the position in the pipe and time. We can then solve $T_w(L, \tau) = T_{1/2}$ for τ .

The temperature gradient of the water in the pipe determines how much heat is transferred to the soil at a particular point l per length and time:

$$\frac{d\dot{Q}}{dl}(l, t) = -c_w \rho_w \pi r^2 v \cdot \frac{\partial T_w}{\partial l}(l, t). \quad (1)$$

Here, $\pi r^2 v$ is the volume of water flowing through the cross-section of the pipe per unit time. This heat propagates through the soil, determined by the temperature gradient field. There are two transport processes taking place simultaneously; heat is transferred radially and along the pipe.

The water and the surrounding soil/ the pipe wall are always in thermodynamic equilibrium, i.e. their temperatures are the same. This means that the heat flux along the pipe also has an influence on the temperature of the water. We therefore assume that at each point in time a quasi-stationary equilibrium is established. Then no heat is transported longitudinally (anymore), and the temperature distribution of the water in the pipe and of the soil in a region around the pipe does not change. Thus, given the resulting temperature distribution $T_w(l, t)$ of this equilibrium at time t , the heat that is released by the water according to Equation (1) is transported radially only and used to warm up the soil in the “next” region, thus enlarging the “first” one that surrounds the pipe and in which the temperature remains constant (at least for short times in this equilibrium state).

As the first hint suggests, the radius of the first region, call it R , grows diffusively, at the rate defined by the heat conductivity, specific heat, and density of the soil. Dimensional analysis yields that this rate is given by

$$\frac{dR}{dt} = \frac{K\kappa}{c\rho} \cdot \frac{1}{R},$$

where K is a constant. The factor R^{-1} appears because it makes sense to assume that the rate at which a volume of cold soil is heated up remains constant, i.e.

$$2\pi R \cdot \frac{dR}{dt} \cdot dl = \text{const} \cdot \frac{\kappa dl}{c\rho}.$$

In order to simplify the problem, it is assumed that the radius of this first region is the same everywhere along the pipe, depending only on time:

$$R(t) = \sqrt{\frac{2K\kappa}{c\rho} \cdot t + r^2}.$$

Moreover, in the first region surrounding the pipe, the heat that flows through the area $2\pi\tilde{r}dl$ should be the same as the energy supplied by the water at each point l . Hence,

$$\frac{d\dot{Q}}{dl} = -2\pi\tilde{r}\kappa \cdot \frac{\partial T_s}{\partial r}(\tilde{r})$$

for all $\tilde{r} \in [r, R(t)]$ and at each position l and time t . Here, T_s denotes the temperature of the soil. Taking into account the boundary condition $T_s(r) = T_w$, solving this equation yields the following temperature distribution:

$$T_s(\tilde{r}) = T_w - \frac{1}{2\pi\kappa} \cdot \frac{d\dot{Q}}{dl} \cdot \ln \frac{\tilde{r}}{r}.$$

We assume here that the heat released per unit length is so small that $T_s(R(t)) > T_0$ everywhere along the pipe and at all relevant times $t < \tau$. Since the total power supplied by the water is not greater than $\dot{Q} = c_w\rho_w\pi r^2 v \cdot (T_1 - T_0)$, this means that the pipe must be sufficiently long.

In order to enlarge the first region at the given rate, i.e. to heat the soil at radius $R(t)$ up to temperature $T_s(R(t))$, the following energy is needed per time and length dl :

$$2\pi R(t)\dot{R}(t) \cdot c\rho \cdot (T_s(R(t)) - T_0) = 2\pi K\kappa \cdot (T_s(R(t)) - T_0).$$

As this must be equal to $\frac{d\dot{Q}}{dl}$, we obtain the following equation:

$$\begin{aligned} \frac{d\dot{Q}}{dl}(l, t) &= 2\pi K\kappa \cdot \left(T_w(l, t) - \frac{1}{2\pi\kappa} \cdot \frac{d\dot{Q}}{dl}(l, t) \cdot \ln \left(\sqrt{\frac{2K\kappa}{c\rho r^2} \cdot t + 1} \right) - T_0 \right) \\ &= 2\pi K\kappa \cdot (T_w(l, t) - T_0) - \frac{K}{2} \cdot \frac{d\dot{Q}}{dl}(l, t) \cdot \ln \left(\frac{2K\kappa}{c\rho r^2} \cdot t + 1 \right) \end{aligned}$$

Solving for $\frac{d\dot{Q}}{dl}$ and using Equation (1) gives the following relation:

$$\frac{2\pi K\kappa \cdot (T_w(l, t) - T_0)}{1 + \frac{K}{2} \cdot \ln \left(\frac{2K\kappa}{c\rho r^2} \cdot t + 1 \right)} = -c_w\rho_w\pi r^2 v \cdot \frac{\partial T_w}{\partial l}(l, t).$$

Since $T_w(0, t) = T_1$, the solution of this equation is given by

$$T_w(l, t) = T_0 + (T_1 - T_0) \cdot \exp\left(-\frac{2K\kappa l}{c_w\rho_w r^2 v \cdot \left(1 + \frac{K}{2} \cdot \ln\left(\frac{2K\kappa}{c\rho r^2} \cdot t + 1\right)\right)}\right). \quad (2)$$

Hence, the temperature of the water inside the pipe decreases exponentially with l and the more time passes, the smaller this decrease is. Furthermore, according to this model, right after the water in the pipe starts flowing, the following temperature distribution arises:

$$T_w(l, 0) = T_0 + (T_1 - T_0) \cdot \exp\left(-\frac{2K\kappa l}{c_w\rho_w r^2 v}\right).$$

This is consistent with the fact that the heat propagates much faster along the pipe than radially, as the first hint also states.

It may occur that the temperature is already greater than $T_{1/2}$ everywhere at the beginning, i.e. that $T_w(L, 0) > T_{1/2}$. This is the case if

$$\frac{2K\kappa L}{c_w\rho_w r^2 v} < \ln 2.$$

Then, $\tau = 0$. Otherwise, using Equation (2), we obtain the following estimate for the waiting time:

$$\tau = \frac{c\rho r^2}{2K\kappa} \cdot \left[\exp\left(\frac{4\kappa L}{\ln 2 \cdot c_w\rho_w r^2 v} - \frac{2}{K}\right) - 1 \right].$$