# Physics Cup 2024, Problem 5

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The qualitative description of the situation in the problem is as follows. As warm water from the boiler flows into the pipe at the temperature $T_1$, because of the lower temperature of the surrounding soil initially at $T_0$, heat escapes to the soil. The dimensions of the pipe and the flow speed (as well as the material properties) are such that the water temperature drops significantly by the time it reaches the tap. Gradually, the surrounding soil heats up which slows the rate of heat loss and the temperature at the tap increases.

The governing equation of heat transfer in the soil is

$$\frac{\partial T_S}{\partial t} = \alpha \nabla^2 T_S = \alpha \left( \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial T_S}{\partial s} \right) + \frac{\partial^2 T_S}{\partial z^2} \right) \tag{1}$$

where $T_S(s, z, t)$ is the temperature of the soil, $s$ is the radial coordinate, $z$ is the position along the pipe, $t$ is the time and $\alpha = \kappa/\rho c$ is the thermal diffusivity of the soil.

We know that the time need to increase the exit temperature of the water to $(T_1 + T_0)/2$ is significantly longer than the time need for a parcel of liquid to travel from the boiler to the tap $L/v$ which means that during the time the parcel is in the pipe, the temperature profile of the water can be taken to be constant. In other words, if the temperature of the water as a function of position along the pipe $z$ and time $t$ is $T_W(z, t)$ and the speed of the water is $v$, then

$$v \frac{\partial T_W}{\partial z} \gg \frac{\partial T_W}{\partial t} \tag{2}$$

The total heat lost by a parcel of fluid of length $\delta z$ which exits the boiler at time $t_0$ by the time $t_0 + t'$ can be obtained by integrating the rate of heat loss to soil through the wall of the pipe which is proportional to the derivative of the temperature of the soil at the wall:

$$\int_{t_0}^{t_0+t'} (-\kappa) \frac{\partial T_S}{\partial s}(r, vt, t_0 + t) 2r\pi \, \delta z \, \mathrm{d}t = -\rho_w c_w r^2 \pi \left( T_W(t_0 + t', vt') - T_0 \right) \tag{3}$$

where $\rho_w$ is the density of the water, $c_w$ is its specific heat capacity and $r$ is the radius of the pipe. By taking the derivative of this expression with respect to time $t'$, we obtain the relationship

$$\frac{\partial T_S}{\partial s} = \frac{\rho_w c_w r}{2\kappa} \left( \frac{\partial T_W}{\partial t} + v \frac{\partial T_W}{\partial z} \right) \approx \frac{\rho_w c_w r v}{2\kappa} \frac{\partial T_W}{\partial z} \tag{4}$$

for all $z$ and $t$ and where we used the approximation (2).

In the preceding discussion we disregarded any heating of the parcel due to conduction in the pipe. It is expected that heat flow along the pipe is mostly negligible because we are interested in the long term behavior and not the comparatively brief transient period. We expect the variation of the temperature along the direction of the pipe to be gradual enough that conduction is negligible. The soil in direct contact with the pipe and the water are of the same temperature (as functions of $z$) and the temperature profile of the soil is going to be almost stationary close to the pipe. Meaning, the heating of the soil close enough to the pipe is going to be slow enough that we can disregard it. Then, since heat flows radially and along the pipe ($z$ direction), both terms on the RHS of equation (1) are positive and therefore small. In other words, heat conduction along the pipe is small with respect to radial heat loss.

Next we have to find the derivative of the temperature of the soil at the wall. We are going to estimate the total amount of heat the soil has received by the size and temperature profile of the quasi-stationary region. While regarding each slice of the soil of width $\delta z$ separately, we employ the condition of stationarity — that the heat flow radially into an annular region through its inner surface of radius $s$ be equal to the heat flow out of the region trough its outer surface radius $s + \mathrm{d}s$. Or equivalently, that

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T_S}{\partial s}\right) = 0 \tag{5}$$

This yields

$$T_S = C_1 \ln s + C_2 \tag{6}$$

where $C_1$ and $C_2$ are some functions of $z$ and $t$. Obviously $C_1$ is negative and for a certain radius $d$, $T_S$ is going to reach $T_0$ with a finite derivative. This, of course, cannot happen in reality so this is where $T_S$ deviates from this expression and the soil experiences more heating but further out there is again no heating. In conjunction with the conditions $T_S(r) = T_W$ and $T_S(d) = T_0$, we have

$$T_S = -\frac{T_W - T_0}{\ln x}\ln\frac{s}{r} + T_W \tag{7}$$

at some time $t$ and distance from the boiler $z$ and where we introduced $x = d/r$ as the parameter determining the size of the region. The estimated total heat received by the slice of soil is

$$\delta Q = 2\pi\rho c\,\delta z\int_r^d (T_S - T_0)s\mathrm{d}s = \pi\rho c r^2\,\delta z\,(T_W - T_0)f(x) \tag{8}$$

where the function $f(x)$ is

$$f(x) = \frac{x^2 - 1}{2\ln x} - 1 \tag{9}$$

On the other hand, the heating power which the slice receives from the pipe is

$$\delta P = 2\pi r\,\delta z\,(-\kappa)\frac{\partial T_S}{\partial s}(r) = 2\pi\,\delta z\,\kappa\frac{T_W - T_0}{\ln x} \tag{10}$$

If we equate $\frac{\mathrm{d}}{\mathrm{d}t}(\delta Q) = \delta P$ while regarding the temperature $T_W$ to be essentially constant we get the following equation:

$$f'(x)\ln x\dot{x} = \frac{2\kappa}{\rho c r^2} \tag{11}$$

In order to simplify, we notice that

$$f'(x)\ln x = x - \frac{x^2 - 1}{2x\ln x} \approx x - 1 \tag{12}$$

This holds as a rough approximation because the expression is 0 for $x = 1$, the derivative of the second term is equal to 0 at $x = 1$, the second term is relatively small compared to $x$ for $x$

somewhat larger than 1 and the second derivative of the expression is very small for all $x$ larger than 1.

Integrating gives

$$x = 1 + \sqrt{\frac{4\kappa}{\rho c r^2} t} \tag{13}$$

This expression depends on neither the temperature of the pipe nor the position $z$ and so the extent of the almost stationary region is roughly the same everywhere along the pipe. From (10) we immediately get the derivative of the temperature of the soil at the pipe wall which by equation (4) gives

$$\frac{\partial T_W}{\partial z} = \frac{-2\kappa (T_W - T_0)}{\rho_w c_w r^2 v \ln\left(1 + \sqrt{\frac{4\kappa}{\rho c r^2} t}\right)} \tag{14}$$

This equation, regarded as an ordinary differential equation for the pipe temperature along $z$, has a simple solution:

$$T_W = (T_1 - T_0) \exp\left\{ \frac{-2\kappa z}{\rho_w c_w r^2 v \ln\left(1 + \sqrt{\frac{4\kappa}{\rho c r^2} t}\right)} \right\} + T_0 \tag{15}$$

Here we used the condition that $T_W = T_1$ at $z = 0$. Setting $T_W(z = L) = (T_1 + T_0)/2$ gives the time necessary for the temperature at the tap to reach $(T_1 + T_0)/2$:

$$t = \frac{\rho c r^2}{4\kappa} \left[ \exp\left\{ \frac{2\kappa L}{\rho_w c_w r^2 v \ln 2} \right\} - 1 \right]^2 \tag{16}$$

This can be further simplified if the exponential in the brackets has to be a lot larger than one. From the condition that $t \gg L/v$ we arrive at

$$\exp\left\{ \frac{2\kappa L}{\rho_w c_w r^2 v \ln 2} \right\} \gg 1 + \sqrt{\frac{4\kappa L}{\rho c r^2 v}} \tag{17}$$

but since, in reality, $\rho c$ and $\rho_w c_w$ are of the same order, the exponential must be a lot larger than one. The estimate for time necessary for the arrival of warm water at the tap is then

$$t = \frac{\rho c r^2}{4\kappa} \exp\left\{ \frac{4\kappa L}{\rho_w c_w r^2 v \ln 2} \right\} \tag{18}$$