Physics Cup — TalTech 2024 — Problem 5

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A water pipe of length L and of internal radius r runs under ground and is surrounded over its entire length by soil at temperature T_0. The specific heat of the soil is c, the density is ρ, and the heat conductance is κ. The characteristics of the pipe walls are identical to those of the soil. A boiler supplying water at a constant temperature $T_1 > T_0$ is attached to the inlet of the pipe, and at the moment of time $t = 0$, a tap is opened at the outlet of the pipe. The water in pipe starts flowing at a constant speed v. How long one has to wait for the water flowing out of the tap to become warm if it is known that this waiting time is significantly bigger than L/v? We call water warm if its temperature is higher than $\frac{1}{2}(T_0 + T_1)$. Only an estimate of the answer is required: you need to provide the correct functional dependence of the waiting time while estimating the magnitude of its parameters. The specific heat of water is c_w, and its density is ρ_w; the water flow in the pipe is turbulent so that the water temperature can be assumed to be constant over a cross-section of the pipe.

First of all, I consider a warm "sausage" of around the pipe, of a fix length l and of a radius r_b. I expect a quasi-stationary temperature profile in this region, so the heat equation in the case of cylindrical symmetry is

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(\kappa R\frac{\mathrm{d}T}{\mathrm{d}R}\right) = 0 \Rightarrow T(R) = T_b - f\ln\left(\frac{R}{r}\right), \quad (1)$$

where $T_b = T(r)$, f is the constant (no dependence on R) defined by the equation

$$T(r_b) = T_0 \Rightarrow T_b - T_0 = f\ln\left(\frac{r_b}{r}\right). \quad (2)$$

Let me calculate the heat accumulated in the "sausage":

$$Q = \int_r^{r_b} (T(R) - T_0)c\rho l \cdot 2\pi R\,\mathrm{d}R. \quad (3)$$

Denote $\alpha = R/r$ and $\beta = r_b/r$ and make a variable change:

$$Q = \pi c\rho l r^2\left((T_b - T_0)(\beta^2 - 1) - f\int_1^\beta 2\alpha\ln\alpha\,\mathrm{d}\alpha\right) =$$
$$= \pi c\rho l r^2\left((T_b - T_0)(\beta^2 - 1) - f\left(\beta^2\ln\beta - \frac{\beta^2 - 1}{2}\right)\right). \quad (4)$$

Let substitute f from Eq. 2,

$$Q = \pi c\rho l r^2 (T_b - T_0)\left(\frac{\beta^2 - 1}{2\ln\beta} - 1\right). \quad (5)$$

Now relate radius r and time t from the process beginning

$$\frac{\partial Q}{\partial\beta}\cdot\dot\beta = qS = 2\pi l\kappa f \Rightarrow$$
$$\left(\beta - \frac{\beta^2 - 1}{2\beta\ln\beta}\right)\dot\beta = \frac{2\kappa}{c\rho r^2}. \quad (6)$$

Neglect the second term assuming $\beta \gg 1$. Now I can find the dependence $\beta(t)$:

$$\beta(t) = \sqrt{\frac{4\kappa t}{c\rho r^2}}. \quad (7)$$

Apart from this consider the case $\beta \gtrsim 1$. Interestingly, in the vicinity of $\beta = 1$ the equation takes on the form (simplify the left part of Eq. 8 using the Taylor series about $\beta = 1$):

$$(\beta - 1)\,\dot\beta = \frac{2\kappa}{c\rho r^2} \Rightarrow \beta(t) = 1 + \sqrt{\frac{4\kappa t}{c\rho r^2}}. \quad (8)$$

Now consider the original task. $l = l(t)$ — the dependence of the "sausage" length on time. The heat given by hot water until time t:

$$Q(t) \approx \pi c_w\rho_w r^2 vt(T_1 - T_0), \quad (9)$$

where we assume that all water until time t leaks at temperature T_0[1]. Compare this equation to Eq. 5. Denote a required time by τ, then $L = l(\tau)$. If $\beta \gg 1$,

$$\frac{\beta^2 - 1}{2\ln\beta} - 1 \sim \frac{\beta^2}{2\ln\beta} \Rightarrow$$
$$\Rightarrow \pi c\rho L r^2 (T_b - T_0)\frac{\beta^2}{2\ln\beta} = \pi c_w\rho_w r^2 v\tau(T_1 - T_0) \Rightarrow$$
$$\Rightarrow \tau = \frac{\rho c r^2}{4\kappa}\exp\left(\frac{4\kappa L}{v\rho_w c_w r^2}\right). \quad (10)$$

Here we have assumed that $T_b = T_1$ and used the Eq. 7. The assumption $\beta \gg 1$ is equal to

$$\exp\left(\frac{4\kappa L}{v\rho_w c_w r^2}\right) \gg 1. \quad (11)$$

If β is about 1:

$$\frac{\beta^2 - 1}{2\ln\beta} - 1 \sim \beta - 1 \Rightarrow$$
$$\Rightarrow \pi c\rho L r^2 (T_b - T_0)\sqrt{\frac{4\kappa\tau}{c\rho r^2}} = \pi c_w\rho_w r^2 v\tau(T_1 - T_0) \Rightarrow$$
$$\Rightarrow \tau = \frac{4\kappa c\rho L^2}{c_w^2\rho_w^2 v^2 r^2}. \quad (12)$$

Answer.

$$\tau = \begin{cases} \frac{\rho c r^2}{4\kappa}\exp\left(\frac{4\kappa L}{v\rho_w c_w r^2}\right), & \text{if } \frac{L}{v} \gtrsim \frac{\rho_w c_w r^2}{\kappa}, \\ \frac{4\kappa c\rho L^2}{c_w^2\rho_w^2 v^2 r^2}, & \text{else.} \end{cases} \quad (13)$$

[1]Here I have used that the water coming out is not hot but only warm in the end of the experiment.