2025 Physics Cup Problem 5

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Solution

Lemma 1. *The velocity vector of the satellite traces a circle in velocity space.*

Proof. Place the planet at the origin. The swept area A and angle θ of the position vector r are related by $\mathrm{d}A = \frac{1}{2}r^2\mathrm{d}\theta$. By Newton's law of gravitation,

$$\begin{aligned}\mathrm{d}\mathbf{v} &= -\frac{GM}{r^2}\hat{\mathbf{r}}\mathrm{d}t \\ &= -\frac{GM}{r^2}\hat{\mathbf{r}}\frac{\mathrm{d}A}{L} \\ &= -\frac{GM}{2L}\hat{\mathbf{r}}\mathrm{d}\theta = -\frac{GM}{2L}\langle\cos\theta, \sin\theta\rangle\mathrm{d}\theta,\end{aligned}$$

where we used Kepler's second law of the form $L = \frac{\mathrm{d}A}{\mathrm{d}t}$. Integrating, we get

$$\mathbf{v} = \mathbf{v_0} - \frac{GM}{2L}\langle\sin\theta, -\cos\theta\rangle,$$

which traces a circle, as desired. We call this circle the *velocity circle* □

The velocity circle Γ of the satellite and the points A, B, C in velocity space are shown in Figure 1. By the Pythagorean Theorem, we find $AB = \sqrt{v_1^2 + v_2^2}$ and $BC = \sqrt{v_2^2 + v_3^2}$. Thus, the radius of Γ is given by

$$R = \frac{abc}{4[ABC]} = \frac{(v_1 + v_3)\sqrt{v_1^2 + v_2^2}\sqrt{v_2^2 + v_3^2}}{2v_1v_2 + 2v_2v_3} = \frac{\sqrt{(v_1^2 + v_2^2)(v_2^2 + v_3^2)}}{2v_2}.$$

Let $v_{\min}$ and $v_{\max}$ be the minimum and maximum velocities of the satellite, respectively. By conservation of angular momentum, these velocities occur at the apoapsis and periapsis, with $v_{\min}r_{\max} = v_{\max}r_{\min}$. Hence, the eccentricity of the orbit is given by

$$e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} = \frac{v_{\max} - v_{\min}}{v_{\max} + v_{\min}} = \frac{2d}{2R} = \frac{d}{R}.$$

Lemma 2 (Power of a Point)**.** $R^2 - d^2 = v_1v_3$

Proof. Note that

$$\triangle AXC' \sim \triangle A'XC \implies \frac{v_1}{R - d} = \frac{R + d}{v_3}.$$

Rearranging gives the desired result. □

Hence, we can directly find

$$\frac{d}{R} = \sqrt{1 - \frac{v_1v_3}{R^2}} = \boxed{\sqrt{1 - \frac{4v_1v_2^2v_3}{(v_1^2 + v_2^2)(v_2^2 + v_3^2)}}}.$$

For the given values in the problem, the eccentricity is $\sqrt{\frac{17}{65}} \approx 0.5114$.

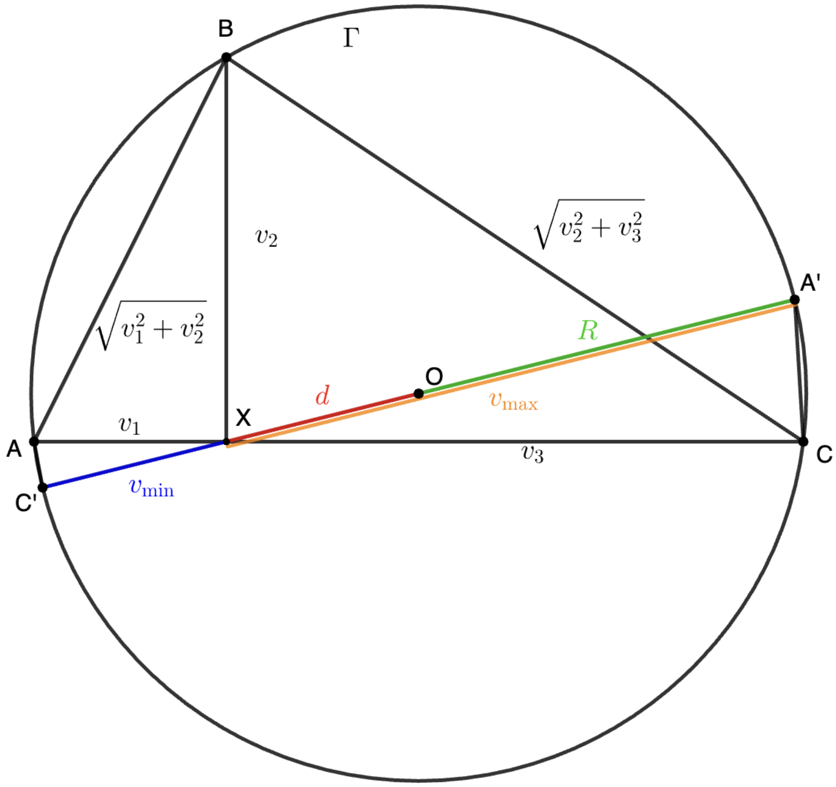

Figure 1: Diagram of the velocity circle Γ. The origin is located at X.